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Abstract. We study a fourth order geometric evolution problem on a network of curves in a bounded domain Ω. The flow decreases a weighted total length of the curves and preserves the enclosed volumes. Stationary solutions of the flow are critical points of a partition problem in Ω. In this paper we study the linearized stability of stationary solutions using the H^{-1}-gradient flow structure of the problem. Important issues are the development of an appropriate PDE formulation of the geometric problem and Poincaré type estimate on a network of curves.

Key words. fourth order geometric evolution problem, surface diffusion, network of curves, linearized stability, Poincaré inequality on a network, H^{-1}-gradient flow

AMS subject classifications. 35B35, 35G30, 35K55, 35R35, 53C44.

1 Introduction

Motion by surface diffusion

$$V = -\Delta_s \kappa \tag{1.1}$$

is a fourth order geometric evolution for an evolving hypersurface $\Gamma = \{\Gamma_t\}_{t\geq 0}$ that has the property that the perimeter of the enclosed volume decreases whereas the volume is conserved. The latter is in contrast to the second order motion by mean curvature $V = \kappa$ where also the perimeter decreases but the volume is not conserved. In the above, V is the normal velocity of the surface, κ is the sum of the principal curvatures of the surface, and Δ_s is the Laplace-Beltrami operator of the surface.

The surfaces with constant mean curvature are stationary solutions of (1.1). A natural question to ask is whether these solutions are stable under (1.1). This question has been answered positive by Elliott and Garcke [2] for circles in the plane and by Escher, Mayer and Simonett [4] for spheres in higher dimensions.

If a hypersurface lies in a bounded domain Ω and is attached to the outer boundary, the surface diffusion has to take care of boundary conditions. Natural boundary conditions are a 90° angle condition and a no-flux condition, i.e. we require on $\Gamma \cap \partial\Omega$

$$\text{a } 90^\circ \text{ angle condition,} \tag{1.2}$$

$$\tau \cdot \nabla_s \kappa = 0, \tag{1.3}$$

where ∇_s is the surface gradient and τ is the outer conormal of Γ at its boundary points. We remark that we assume that $\partial\Gamma$ is contained in $\partial\Omega$. For this evolution law, a linearized stability criterion for spherical arcs that attach to the boundary with a 90° angle condition has been given by the authors [7]. For the mean curvature flow, one can also consider situations where a hypersurface is attached to the outer boundary. In this case, only an angle condition has to be fulfilled. We refer to [5, 6] for a stability analysis in this case.

In many situations of interest, different hypersurfaces moving by surface diffusion meet at junctions. Assume that three evolving hypersurfaces $\Gamma^i = \{\Gamma^i_t\}_{t\geq 0}$ fulfill the surface diffusion equation

$$V^i = -m^i\gamma^i\Delta_s\kappa^i \tag{1.4}$$

for $i = 1, 2, 3$, where γ^i is the surface energy density of the interface i and m^i is the mobility of the interface i. If the three hypersurfaces meet at a triple junction $p(t)$, we require that the following conditions hold:

$$\sphericalangle(\Gamma^1(t), \Gamma^2(t)) = \theta^3, \quad \sphericalangle(\Gamma^2(t), \Gamma^3(t)) = \theta^1, \quad \sphericalangle(\Gamma^3(t), \Gamma^1(t)) = \theta^2, \tag{1.5}$$

$$\gamma^1\kappa^1 + \gamma^2\kappa^2 + \gamma^3\kappa^3 = 0, \tag{1.6}$$

$$m^1\gamma^1\nabla_s\kappa^1 \cdot T^1 = m^2\gamma^2\nabla_s\kappa^2 \cdot T^2 = m^3\gamma^3\nabla_s\kappa^3 \cdot T^3. \tag{1.7}$$

Here the quantity $\sphericalangle(\Gamma^i(t), \Gamma^j(t))$ denotes the angle between $\Gamma^i(t)$ and $\Gamma^j(t)$, and T^i is the inner conormal to $\partial\Gamma^i(t)$ at the triple junction. The angle conditions (1.5) follow from a force balance at the triple junction (Young's law), the second condition (1.6) follows from the continuity of chemical potentials, and the third conditions (1.7) are the flux balance at the triple junction. The angles $\theta^1, \theta^2, \theta^3$ are related through the identity $\theta^1 + \theta^2 + \theta^3 = 2\pi$ and via Young's law which is

$$\frac{\sin\theta^1}{\gamma^1} = \frac{\sin\theta^2}{\gamma^2} = \frac{\sin\theta^3}{\gamma^3}. \tag{1.8}$$

Then we obtain from (1.5) and (1.8) that

$$\gamma^1 T^1 + \gamma^2 T^2 + \gamma^3 T^3 = 0$$

which is the force balance at the triple junction. The condition (1.7) relates the inward pointing parts of the fluxes $m^i\gamma^i\nabla_s\kappa^i$ to each other where ∇_s denotes the surface gradient.

For a network of curves evolving with respect to (1.4) fulfilling (1.5)-(1.7) at the triple junction and (1.2)-(1.3) at contact points with $\partial\Omega$, a computation shows that a network of curves decreases the weighted total length

$$\sum_{i=1}^{3} \gamma^i L[\Gamma^i(t)] \tag{1.9}$$

and preserves the enclosed areas. We refer to Garcke and Novick-Cohen [8] for more details on the model and a verification of the above properties.

From the above properties, it seems natural to expect that surface diffusion for a network of curves leads to solutions which converge for large time to solutions of a partitioning problem, i.e. to a partition that minimizes (1.9) under a volume constraint. Or, to be more precise, we expect the convergence to critical points of (1.9) at large time. These large time limits will have constant curvature (on each curve) and fulfill the angle conditions (1.2) and (1.5). In addition, the three constant curvatures on the arcs have to fulfill (1.6). Such a network is a stationary solution to (1.4), (1.2)-(1.3), (1.5)-(1.7). It is the goal of this paper to derive a stability criterion for stationary solutions to (1.4), (1.2)-(1.3), (1.5)-(1.7). The criterion will be based on a linearized stability consideration. In the papers [5, 6, 10], the authors studied linearized stability for the mean curvature flow, and also the authors of this paper previously studied the case of surface diffusion without triple junctions [7].

Let us briefly outline how we proceed. A first preparatory but important step is to come up with a proper representation of a network of curves. We parameterize curves around a stationary curve with the help of a modified distance function. It is not possible to use distance functions since the triple junction might move and hence we have to introduce a certain tangential adjustment in order to be able to parameterize for all time on a fixed parameter interval. The requirement that all curves meet at a triple point leads to additional difficulties. We then formulate the evolution problem with the help of this parameterization and derive a highly nonlinear problem which will be linearized in Section 3. The linearized problem is complicated but has a gradient flow structure with respect to a certain H^{-1}-inner product on a network of curves. This is the main observation which will greatly simplify the stability analysis. In fact the linearized problem is the gradient flow of the quadratic form related to the second variation of (1.9) and hence one can expect that the stability of solutions depends on the fact whether the stationary solutions is a linearly stable extremum of (1.9).

In order to use spectral theory to analyze the linearized problem, we show self-adjointness of the linearized spatial operator with respect to the H^{-1}-inner product. Further, we show that the spectrum and hence the stability behaviour has a certain monotone dependence on the curvature of the outer boundary and the length of the curves. We then formulate the stability criterion and finally apply the criterion to several specific geometries. In this context, we refer to the proof of the double bubble conjecture [9] where also a stability analysis involving the second variation of configurations with triple junction has been used.

2 Parameterization and PDE formulation

In this section, we derive parameterizations that are convenient to formulate the evolution problem in a PDE context. Let Ω be a bounded domain in $\mathbb{R}^2$ containing $(0,0)$ and having smooth boundary. We assume that Ω and $\partial\Omega$ are given as

$$\Omega = \{x \in \mathbb{R}^2 \,|\, \psi(x) < 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^2 \,|\, \psi(x) = 0\}$$

with a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $x \in \partial\Omega$, i.e. if $\psi(x) = 0$. Let Γ^i_* $(i = 1, 2, 3)$ be straight lines or circular arcs with constant curvature κ^i_* satisfying

$$\gamma^1\kappa^1_* + \gamma^2\kappa^2_* + \gamma^3\kappa^3_* = 0. \tag{2.1}$$

Further, the Γ^i_* $(i = 1, 2, 3)$ are assumed to meet the outer boundary with an angle $\pi/2$ and have $(0,0)$ (without loss of generality) as a common point (triple junction). Then we define an arc-length parameterizations of Γ^i_* $(i = 1, 2, 3)$ as

$$\Gamma^i_* = \{\Phi^i_*(\sigma) \,|\, \sigma \in [0, l^i]\}$$

with $\Phi^i_*(0) = (0,0)$ and $\Phi^i_*(l^i) \in \partial\Omega$, where l^i is the length of Γ^i_*. Note that Φ^i_* can be extended naturally either to a parameterization of a full circle or a straight line. We now introduce a stretched curvilinear coordinate system around the curves Γ^i_* $(i = 1, 2, 3)$. In order to allow for a tangential stretching of the coordinate system close to $\partial\Omega$, we set

$$\mu^i_\Omega(q) := \max\{\sigma \mid \Phi^i_*(\sigma) + qN^i_*(\sigma) \in \Omega\}.$$

Now we define a set of admissible parameterizations with the help of

$$\rho^i : [0, l^i] \to \mathbb{R}, \quad \mu^i \in \mathbb{R} \quad (i = 1, 2, 3)$$

which fulfill

$$\Phi^1_*(\mu^1) + \rho^1(0)N^1_*(\mu^1) = \Phi^2_*(\mu^2) + \rho^2(0)N^2_*(\mu^2) = \Phi^3_*(\mu^3) + \rho^3(0)N^3_*(\mu^3). \tag{2.2}$$

Here the parameter μ^i allows for a tangential movement at the triple junction. In addition, we define

$$\Psi^i(\sigma, q, \mu^i) := \Phi^i_*(\xi^i(\sigma, q, \mu^i)) + qN^i_*(\xi^i(\sigma, q, \mu^i))$$

where

$$\xi^i(\sigma, q, \mu^i) := \mu^i + \frac{\sigma}{l^i}\{\mu^i_\Omega(q) - \mu^i\}.$$

Note that $\xi^i(\sigma, 0, 0) = \sigma$ and $\xi^i(0, q, \mu^i) = \mu^i$. Then we define

$$\Phi^i(\sigma) := \Psi^i(\sigma, \rho^i(\sigma), \mu^i)$$

as the parameterizations of a set of curves Γ^i $(i = 1, 2, 3)$ having the properties that they meet at a triple junction and end on the boundary $\partial\Omega$. Note that

$$T^i = \frac{1}{J^i(\boldsymbol{u}^i)}\Phi^i_\sigma, \quad N^i = \frac{1}{J^i(\boldsymbol{u}^i)}R\Phi^i_\sigma,$$

where $\boldsymbol{u}^i = (\rho^i, \mu^i)$,

$$J^i(\boldsymbol{u}^i) := |\Phi^i_\sigma(\sigma)| = \sqrt{|\Psi^i_\sigma|^2 + 2(\Psi^i_\sigma, \Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma + |\Psi^i_q|^2|\rho^i_\sigma|^2},$$

and R denotes the anti-clockwise rotation by $\pi/2$. We now consider evolving curves

$$\Gamma^i(t) := \{\Phi^i(\sigma, t) \mid \sigma \in [0, l^i]\} \tag{2.3}$$

where the Φ^i are defined for each t via admissible $(\rho^1(\sigma, t), \rho^2(\sigma, t), \rho^3(\sigma, t), \mu^1(t), \mu^2(t), \mu^3(t))$, i.e. we require that (2.2) holds.

We now derive evolution equations for ρ^i and μ^i that have to hold in the case that Γ^i $(i = 1, 2, 3)$ in (2.3) solve (1.4), (1.2)-(1.3), (1.5)-(1.7). The normal velocity V^i of $\Gamma^i(t)$ is given as

$$V^i = (\Phi^i_t, N^i)_{\mathbb{R}^2} = \frac{1}{J^i(\boldsymbol{u}^i)}\Big\{(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}\rho^i_t + (\Psi^i_\mu, N^i)_{\mathbb{R}^2}\mu^i_t\Big\}$$

where

$$N^i = \frac{1}{J^i(\boldsymbol{u}^i)}(R\Psi^i_\sigma + R\Psi^i_q\rho^i_\sigma).$$

In addition, the curvature $\kappa^i(=\kappa^i(\boldsymbol{u}^i))$ of $\Gamma^i(t)$ is computed as

$$\begin{aligned}\kappa^i(\boldsymbol{u}^i) &= \frac{1}{(J^i(\boldsymbol{u}^i))^3}(\Phi^i_{\sigma\sigma}, R\Phi^i_\sigma)_{\mathbb{R}^2}\\ &= \frac{1}{(J^i(\boldsymbol{u}^i))^3}\Big[(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}\rho^i_{\sigma\sigma} + \{2(\Psi^i_{\sigma q}, R\Psi^i_\sigma)_{\mathbb{R}^2} + (\Psi^i_{\sigma\sigma}, R\Psi^i_q)_{\mathbb{R}^2}\}\rho^i_\sigma\\ &\quad + \{(\Psi^i_{qq}, R\Psi^i_\sigma)_{\mathbb{R}^2} + 2(\Psi^i_{\sigma q}, R\Psi^i_q)_{\mathbb{R}^2} + (\Psi^i_{qq}, R\Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma\}(\rho^i_\sigma)^2 + (\Psi^i_{\sigma\sigma}, R\Psi^i_\sigma)_{\mathbb{R}^2}\Big].\end{aligned}$$

Thus the surface diffusion equation (1.4) can be reformulated as

$$\rho^i_t = -m^i\gamma^i a^i(\boldsymbol{u}^i)\Delta(\boldsymbol{u}^i)\kappa^i(\boldsymbol{u}^i) + b^i(\boldsymbol{u}^i)\mu^i_t, \tag{2.4}$$

where

$$\begin{aligned}a^i(\boldsymbol{u}^i) &:= \frac{J^i(\boldsymbol{u}^i)}{(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}}, \quad b^i(\boldsymbol{u}^i) := -\frac{(\Psi^i_\mu, R\Psi^i_\sigma)_{\mathbb{R}^2} + (\Psi^i_\mu, R\Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma}{(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}},\\ \Delta(\boldsymbol{u}^i) &:= \frac{1}{J^i(\boldsymbol{u}^i)}\partial_\sigma\Big(\frac{1}{J^i(\boldsymbol{u}^i)}\partial_\sigma\Big) = \frac{1}{(J^i(\boldsymbol{u}^i))^2}\partial^2_\sigma + \frac{1}{J^i(\boldsymbol{u}^i)}\Big(\partial_\sigma\frac{1}{J^i(\boldsymbol{u}^i)}\Big)\partial_\sigma.\end{aligned}$$

Remark 2.1 The terms $b^i(\boldsymbol{u}^i)\mu^i_t$ will give no contributions to the linearization (see Section 3). However, using (2.2) and (2.4), μ^i_t can be written with the help of $\partial^4_\sigma\rho^i|_{\sigma=0}$ and some lower order differential terms. For the nonlinear problem, it can be shown that these terms are perturbations of the principal part when we study the generator of the analytic semigroup for the equation (2.4).

Let us discuss the boundary conditions at the triple junction. First we have

$$\Phi^1(0) = \Phi^2(0) = \Phi^3(0). \tag{2.5}$$

This is just (2.2). The angle conditions (1.5) are given by

$$(\Phi^1_\sigma, \Phi^2_\sigma)_{\mathbb{R}^2} = |\Phi^1_\sigma||\Phi^2_\sigma|\cos\theta^3, \quad (\Phi^1_\sigma, \Phi^3_\sigma)_{\mathbb{R}^2} = |\Phi^1_\sigma||\Phi^3_\sigma|\cos\theta^2 \tag{2.6}$$

at $\sigma = 0$ with the representation

$$(\Phi^i_\sigma, \Phi^j_\sigma)_{\mathbb{R}^2} = (\Psi^i_\sigma, \Psi^j_\sigma)_{\mathbb{R}^2} + (\Psi^i_\sigma, \Psi^j_q)_{\mathbb{R}^2}\rho^j_\sigma + (\Psi^i_q, \Psi^j_\sigma)_{\mathbb{R}^2}\rho^i_\sigma + (\Psi^i_q, \Psi^j_q)_{\mathbb{R}^2}\rho^i_\sigma\rho^j_\sigma.$$

The condition guaranteeing the continuity of the chemical potentials (1.6) can be restated as

$$\gamma^1\kappa^1(\boldsymbol{u}^1) + \gamma^2\kappa^2(\boldsymbol{u}^2) + \gamma^3\kappa^3(\boldsymbol{u}^3) = 0 \quad \text{at } \sigma = 0. \tag{2.7}$$

The balance of fluxes $m^1\gamma^1\kappa^1_s = m^2\gamma^2\kappa^2_s = m^3\gamma^3\kappa^3_s$ is written as

$$\frac{m^1\gamma^1}{J^1(\boldsymbol{u}^1)}\partial_\sigma\kappa^1(\boldsymbol{u}^1) = \frac{m^2\gamma^2}{J^2(\boldsymbol{u}^2)}\partial_\sigma\kappa^2(\boldsymbol{u}^2) = \frac{m^3\gamma^3}{J^3(\boldsymbol{u}^3)}\partial_\sigma\kappa^3(\boldsymbol{u}^3) \quad \text{at } \sigma = 0. \tag{2.8}$$

Remark 2.2 The equations (2.5)-(2.8) lead to nine conditions at $\sigma = 0$. The three fourth order PDEs for the functions ρ^i $(i = 1, 2, 3)$ in (2.4) need six boundary conditions at $\sigma = 0$, i.e. there are six degrees of freedoms coming from the PDEs in (2.4). We have three additional degrees of freedoms for μ^i $(i = 1, 2, 3)$, so that the number of conditions agree with the number of degrees of freedoms. Thus the number of the boundary conditions at $\sigma = 0$ is appropriate.

On the outer boundary $\partial\Omega$, we obtain that the angle condition (1.2) is equivalent to $(R\Phi^i_\sigma, \nabla\psi(\Phi^i))_{\mathbb{R}^2} = 0$ and hence we have

$$(R\Psi^i_\sigma + R\Psi^i_q\rho^i_\sigma, \nabla\psi(\Psi^i))_{\mathbb{R}^2} = 0 \ \text{ at } \ \sigma = l^i \ \ (i = 1, 2, 3). \tag{2.9}$$

The no-flux condition $\kappa^i_s = 0$ is equivalent to

$$\partial_\sigma\kappa^i(\boldsymbol{u}^i) = 0 \ \text{ at } \ \sigma = l^i \ \ (i = 1, 2, 3). \tag{2.10}$$

3 Linearization

In this section, we linearize the nonlinear boundary value problem stated in Section 2. The functions Ψ^i have the following properties which we need to derive the linearized problem.

Lemma 3.1 *The parameterizations Ψ^i in Section 2 fulfill*

(i) $\Psi^i(\sigma, 0, 0) = \Phi^i_*(\sigma)$ *and* $\Psi^i(\sigma, q, 0) = \Phi^i_*(\sigma\mu^i_\Omega(q)/l^i) + qN^i_*(\sigma\mu^i_\Omega(q)/l^i)$.

(ii) $\Psi^i_\sigma(\sigma, 0, 0) = T^i_*(\sigma)$, $\Psi^i_q(\sigma, 0, 0) = N^i_*(\sigma)$, *and* $\Psi^i_\mu(\sigma, 0, 0) = (1 - \sigma/l^i)T^i_*(\sigma)$.

(iii) $\Psi^i_{\sigma\sigma}(\sigma, 0, 0) = \kappa^i_*N^i_*(\sigma)$, $\Psi^i_{\sigma q}(\sigma, 0, 0) = -\kappa^i_*T^i_*(\sigma)$, *and* $\Psi^i_{\sigma\mu}(\sigma, 0, 0) = (-1/l^i)T^i_*(\sigma) + (1 - \sigma/l^i)\kappa^i_*N^i_*(\sigma)$.

(iv) $\Psi^i_{\sigma\sigma q}(\sigma, 0, 0) = -(\kappa^i_*)^2N^i_*(\sigma)$ *and* $\Psi^i_{\sigma\sigma\mu}(\sigma, 0, 0) = (-2\kappa^i_*/l^i)N^i_*(\sigma) - (1 - \sigma/l^i)(\kappa^i_*)^2T^i_*(\sigma)$.

Proof. By the definition of Ψ^i, (i) is obvious. Let us prove (ii). Differentiating $\Psi^i(\sigma, 0, 0) = \Phi^i_*(\sigma)$ with respect to σ, we readily derive $\Psi^i_\sigma(\sigma, 0, 0) = T^i_*(\sigma)$. Applying a similar argument to [7, Lemma 3.1], we obtain $(\mu^i_\Omega(q))'|_{q=0} = 0$. Thus it follows from (i) that $\Psi^i_q(\sigma, 0, 0) = N^i_*(\sigma)$. Moreover, by the definition of ξ^i, we have

$$\xi^i_\mu(\sigma, 0, 0) = 1 - \sigma/l^i.$$

The definition of Ψ^i and the Frenet-Serret formulas give

$$\Psi^i_\mu(\sigma, q, \mu^i) = \xi_\mu(\sigma, q, \mu^i)(1 - q\kappa^i_*)T^i_*(\xi(\sigma, q, \mu^i)).$$

Setting $(q, \mu^i) = (0, 0)$, the third property of (ii) is derived. Finally, by using (ii) and Frenet-Serret formulas, we have (iii)-(iv). $\square$

Using Lemma 3.1, we observe $b^i(\mathbf{0}) = 0$, $J^i(\mathbf{0}) = 1$, and

$$\partial J^i(\mathbf{0})[\boldsymbol{\eta}^i] = -\kappa^i_*v^i - \frac{1}{l^i}\tau^i \tag{3.1}$$

where $\partial J^i(\mathbf{0})[\boldsymbol{\eta}^i]$ is the Fréchet derivative of J^i at $\mathbf{0} = (0, 0)$ and $\boldsymbol{\eta}^i = (v^i(\cdot), \tau^i) \in C^1([0, l^i]) \times \mathbb{R}$.

Let us derive the linearization of (2.4). We define the operator

$$F^i(\boldsymbol{u}^i) := \rho^i_t - m^i\gamma^iG^i(\boldsymbol{u}^i) - b^i(\boldsymbol{u}^i)\mu^i_t,$$

which maps functions $\boldsymbol{u}^i = (\rho^i(\cdot, \cdot), \mu^i(\cdot)) \in C^{4,1}(Q^i_T) \times C^1([0, T])$ to functions in $C^{0,0}(Q^i_T) \times C^0([0, T])$, where $Q^i_T := [0, l^i] \times [0, T]$ and

$$G^i(\boldsymbol{u}^i) := -a^i(\boldsymbol{u}^i)\Delta(\boldsymbol{u}^i)\kappa^i(\boldsymbol{u}^i).$$

Then the equation (2.4) is represented as $F^i(\boldsymbol{u}^i) = 0$. We derive the Fréchet derivative of F^i at $\mathbf{0} = (0, 0)$ in the following lemma.

Lemma 3.2 *The operator F^i is Fréchet differentiable with the derivative*

$$\partial F^i(\mathbf{0})[\boldsymbol{\eta}^i] = v^i_t + m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma},$$

where $\partial F^i(\mathbf{0})$ is the Fréchet derivative of F^i at $\mathbf{0} = (0,0)$ and $\boldsymbol{\eta}^i = (v^i(\cdot,\cdot), \tau^i(\cdot)) \in C^{4,1}(Q^i_T) \times C^1([0,T])$.

Using Lemma 3.1 and taking into account $b^i(\mathbf{0}) = 0$, the proof of this lemma is similar to [7, the proof of Lemma 3.2]. Thus we omit it.

Let us linearize the boundary conditions (2.5)-(2.10). We first derive the following lemma from (2.5).

Lemma 3.3 *Let $(v^i(\cdot), \tau^i)$ $(i = 1,2,3)$ be in $C^0([0,l^i]) \times \mathbb{R}$. Then we obtain the following conditions at $\sigma = 0$:*

(i) $\gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0$ *at $\sigma = 0$.*

(ii) For $i,j,k \in \{1,2,3\}$ mutually different

$$\tau^i = \frac{1}{s^i}(c^j v^j - c^k v^k) \quad \text{at} \quad \sigma = 0,$$

where $c^i := \cos\theta^i$ and $s^i := \sin\theta^i$.

Proof. Let us prove (i). For $\boldsymbol{u}^i = (\rho^i(\cdot), \mu^i) \in C^0([0,l^i]) \times \mathbb{R}$, we set

$$B_0(\boldsymbol{u}^i, \boldsymbol{u}^j) := \Phi^i_*(\mu^i) + \rho^i(0)N^i_*(\mu^i) - \Phi^j_*(\mu^j) - \rho^j(0)N^j_*(\mu^j).$$

Then (2.5) is represented as $B_0(\boldsymbol{u}^i, \boldsymbol{u}^j) = 0$ at $\sigma = 0$. Computing the Fréchet derivative of $B_0(\boldsymbol{u}^i, \boldsymbol{u}^j) = 0$, we derive

$$0 = \partial B_0(\mathbf{0},\mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j] = \tau^i T^i_*(0) + v^i(0)N^i_*(0) - \tau^j T^j_*(0) - v^j(0)N^j_*(0),$$

where $\partial B_0(\mathbf{0},\mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j]$ is the Fréchet derivative of B_0 at $(\mathbf{0},\mathbf{0})$ and $\boldsymbol{\eta}^i = (v^i(\cdot), \tau^i)$. This implies that

$$\tau^i T^i_*(0) + v^i(0)N^i_*(0) = \tau^j T^j_*(0) + v^j(0)N^j_*(0). \tag{3.2}$$

Set $p_* := \tau^1 T^1_*(0) + v^1(0)N^1_*(0) = \tau^2 T^2_*(0) + v^2(0)N^2_*(0) = \tau^3 T^3_*(0) + v^3(0)N^3_*(0)$. Then we obtain $(p_*, N^i_*(0))_{\mathbb{R}^2} = v^i(0)$ $(i = 1,2,3)$, so that Young's law for the stationary curves Γ^i_* $(i = 1,2,3)$ gives

$$\sum_{i=1}^{3} \gamma^i v^i(0) = \sum_{i=1}^{3} \gamma^i (p_*, N^i_*(0))_{\mathbb{R}^2} = \big(p_*, \sum_{i=1}^{3} \gamma^i N^i_*(0)\big)_{\mathbb{R}^2} = 0.$$

We prove (ii). By means of (3.2), we see

$$\tau^i = \tau^j (T^i_*(0), T^j_*(0))_{\mathbb{R}^2} + v^j(0)(T^i_*(0), N^j_*(0))_{\mathbb{R}^2}.$$

Then it follows from the angle conditions for the stationary curves Γ^i_* at p_* that

$$(T^i_*(0), T^j_*(0))_{\mathbb{R}^2} = \cos\theta^k, \quad (T^i_*(0), N^j_*(0))_{\mathbb{R}^2} = -\sin\theta^k$$

for $i, j, k \in \{1,2,3\}$ mutually different, so that we derive

$$\tau^i = \tau^j \cos\theta^k - v^j(0)\sin\theta^k.$$

Setting $c^i := \cos\theta^i$ and $s^i := \sin\theta^i$, we have

$$(1 - c^i c^j c^k)\tau^i = -\big\{c^k c^i s^j v^i(0) + s^k v^j(0) + c^k s^i v^k(0)\big\}.$$

Further, (1.8) and (i) imply

$$(1 - c^i c^j c^k)\tau^i = -\frac{1}{s^i}\Big[\big\{(s^k s^i - c^k c^i (s^j)^2\big\}v^j(0) + \big\{c^k (s^i)^2 - c^k c^i s^j s^k\big\}v^k(0)\Big]$$

Since we observe

$$s^k s^i - c^k c^i (s^j)^2 = -c^j(1 - c^i c^j c^k), \quad c^k (s^i)^2 - c^k c^i s^j s^k = c^k(1 - c^i c^j c^k),$$

we are led to the desired result. $\square$

Lemma 3.4 *Let $v^i(\cdot)$ $(i = 1,2,3)$ be in $C^1([0,l^i])$. Then, for $i, j, k \in \{1,2,3\}$ mutually different, the linearization of the angle conditions (2.6) are*

$$\frac{1}{s^i}(c^j \kappa_*^j - c^k \kappa_*^k)v^i + v_\sigma^i = \frac{1}{s^j}(c^k \kappa_*^k - c^i \kappa_*^i)v^j + v_\sigma^j \quad at \quad \sigma = 0,$$

where $c^i := \cos\theta^i$ and $s^i := \sin\theta^i$.

Proof. For $\boldsymbol{u}^i = (\rho^i(\cdot), \mu^i) \in C^1([0,l^i]) \times \mathbb{R}$, we set

$$B_1(\boldsymbol{u}^i, \boldsymbol{u}^j) := b_1(\boldsymbol{u}^i, \boldsymbol{u}^j) - J^i(\boldsymbol{u}^i)J^j(\boldsymbol{u}^j)\cos\theta^k,$$

where

$$b_1(\boldsymbol{u}^i, \boldsymbol{u}^j) := (\Psi_\sigma^i, \Psi_\sigma^j)_{\mathbb{R}^2} + (\Psi_\sigma^i, \Psi_q^j)_{\mathbb{R}^2}\rho_\sigma^j + (\Psi_q^i, \Psi_\sigma^j)_{\mathbb{R}^2}\rho_\sigma^i + (\Psi_q^i, \Psi_q^j)_{\mathbb{R}^2}\rho_\sigma^i \rho_\sigma^j.$$

Then the boundary conditions for the angles are rewritten as $B_1(\boldsymbol{u}^i, \boldsymbol{u}^j) = 0$ at $\sigma = 0$, which gives

$$\partial B_1(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j] = 0 \quad \text{at} \quad \sigma = 0, \tag{3.3}$$

where $\partial B_1(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j]$ is the Fréchet derivative of B_1 at $(\mathbf{0}, \mathbf{0})$ and $\boldsymbol{\eta}^i = (v^i(\cdot), \tau^i) \in C^1([0,l^i]) \times \mathbb{R}$. Let us derive the Fréchet derivative $\partial B_1(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j]$. Using Lemma 3.1, the Fréchet derivative of $\Psi_\sigma^i(0,\cdot,\cdot)$ at $\mathbf{0} = (0,0)$ is

$$\partial\Psi_\sigma^i(0,\cdot,\cdot)(\mathbf{0})[\boldsymbol{\eta}^i] = -\kappa_*^i T_*^i(0)v^i(0) + \Big(-\frac{1}{l^i}T_*^i(0) + \kappa_*^i N_*^i(0)\Big)\tau^i.$$

This and Lemma 3.1(ii) imply that

$$\begin{aligned}
&\partial b_1(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j] \\
&= -\kappa_*^i (T_*^i(0), T_*^j(0))_{\mathbb{R}^2} v^i(0) - \frac{1}{l^i}(T_*^i(0), T_*^j(0))_{\mathbb{R}^2}\tau^i + \kappa_*^i (N_*^i(0), T_*^j(0))_{\mathbb{R}^2}\tau^i \\
&\quad -\kappa_*^j (T_*^i(0), T_*^j(0))_{\mathbb{R}^2} v^j(0) - \frac{1}{l^j}(T_*^i(0), T_*^j(0))_{\mathbb{R}^2}\tau^j + \kappa_*^j (T_*^i(0), N_*^j(0))_{\mathbb{R}^2}\tau^j \\
&\quad +(T_*^i(0), N_*^j(0))_{\mathbb{R}^2} v_\sigma^j(0) + (N_*^i(0), T_*^j(0))_{\mathbb{R}^2} v_\sigma^i(0),
\end{aligned}$$

where $\partial b_1(\mathbf{0},\mathbf{0})[\boldsymbol{\eta}^i,\boldsymbol{\eta}^j]$ is the Fréchet derivative of b_1 at $(\mathbf{0},\mathbf{0})$. The angle conditions for the stationary curves Γ_*^i give

$$(T_*^i(0),T_*^j(0))_{\mathbb{R}^2}=\cos\theta^k,\quad (T_*^i(0),N_*^j(0))_{\mathbb{R}^2}=-\sin\theta^k,\quad (N_*^i(0),T_*^j(0))_{\mathbb{R}^2}=\sin\theta^k$$

for $i,j,k\in\{1,2,3\}$ mutually different, so that

$$\begin{aligned}\partial b_1(\mathbf{0},\mathbf{0})[\boldsymbol{\eta}^i,\boldsymbol{\eta}^j]=&-\kappa_*^i v^i(0)\cos\theta^k-\frac{1}{l^i}\tau^i\cos\theta^k-\kappa_*^j v^j(0)\cos\theta^k-\frac{1}{l^j}\tau^j\cos\theta^k\\&+(\kappa_*^i\tau^i-\kappa_*^j\tau^j)\sin\theta^k+(v_\sigma^i(0)-v_\sigma^j(0))\sin\theta^k.\end{aligned}$$

Then it follows from $J^i(\mathbf{0})=1$ and (3.1) that

$$\partial B_1(\mathbf{0},\mathbf{0})[\boldsymbol{\eta}^i,\boldsymbol{\eta}^j]=\{(\kappa_*^i\tau^i-\kappa_*^j\tau^j)+(v_\sigma^i(0)-v_\sigma^j(0))\}\sin\theta^k.$$

Thus (3.3) is equivalent to

$$\{(\kappa_*^i\tau^i-\kappa_*^j\tau^j)+(v_\sigma^i(0)-v_\sigma^j(0))\}\sin\theta^k=0.$$

Since $0<\theta^k<\pi$, we derive

$$\kappa_*^i\tau^i+v_\sigma^i(0)=\kappa_*^j\tau^j+v_\sigma^j(0).$$

Recalling Lemma 3.3(ii), we have

$$\frac{\kappa_*^i}{s^i}\{c^jv^j(0)-c^kv^k(0)\}+v_\sigma^i(0)=\frac{\kappa_*^j}{s^j}\{c^kv^k(0)-c^iv^i(0)\}+v_\sigma^j(0)\tag{3.4}$$

for $i,j,k\in\{1,2,3\}$ mutually different. Taking into account (1.8) and Lemma 3.3(i), we observe

$$\begin{aligned}\frac{1}{s^i}\{c^jv^j(0)-c^kv^k(0)\}&=\frac{1}{\gamma^i}\Big\{\frac{c^j}{s^j}\gamma^jv^j(0)-\frac{c^k}{s^k}\gamma^kv^k(0)\Big\}\\&=\frac{1}{\gamma^i}\Big[\frac{c^j}{s^j}\gamma^jv^j(0)+\frac{c^k}{s^k}\{\gamma^iv^i(0)+\gamma^jv^j(0)\}\Big]\\&=\frac{1}{\gamma^i}\Big\{\frac{c^k}{s^k}\gamma^iv^i(0)+\Big(\frac{c^j}{s^j}+\frac{c^k}{s^k}\Big)\gamma^jv^j(0)\Big\}\\&=\frac{1}{s^k}\{c^kv^i(0)-v^j(0)\},\end{aligned}$$

Applying an analogous argument, we also have

$$\frac{1}{s^j}\{c^kv^k(0)-c^iv^i(0)\}=\frac{1}{s^k}\{v^i(0)-c^kv^j(0)\}.$$

Putting these terms into (3.4), we obtain

$$\frac{\kappa_*^i}{s^k}\{c^kv^i(0)-v^j(0)\}+v_\sigma^i(0)=\frac{\kappa_*^j}{s^k}\{v^i(0)-c^kv^j(0)\}+v_\sigma^j(0).$$

This implies that

$$\frac{1}{s^k}(c^k\kappa_*^i-\kappa_*^j)v^i(0)+v_\sigma^i(0)=\frac{1}{s^k}(\kappa_*^i-c^k\kappa_*^j)v^j(0)+v_\sigma^j(0).$$

Using (1.8) and (2.1), we derive

$$\begin{aligned}
\frac{1}{s^k}(c^k\kappa_*^i-\kappa_*^j) &= -\frac{c^k}{\gamma^k s^i}(\gamma^j\kappa_*^j+\gamma^k\kappa_*^k)-\frac{1}{s^k}\kappa_*^j \\
&= -\frac{1}{s^k s^i}(s^j c^k+s^i)\kappa_*^j-\frac{c^k}{s^i}\kappa_*^k \\
&= -\frac{1}{s^k s^i}\{s^j c^k-(s^j c^k+c^j s^k)\}\kappa_*^j-\frac{c^k}{s^i}\kappa_*^k \\
&= \frac{1}{s^i}(c^j\kappa_*^j-c^k\kappa_*^k).
\end{aligned}$$

Also, we see

$$\frac{1}{s^k}(\kappa_*^i-c^k\kappa_*^j)=\frac{1}{s^j}(c^k\kappa_*^k-c^i\kappa_*^i).$$

This completes the proof. $\square$

Lemma 3.5 *Let $v^i(\cdot)$ $(i=1,2,3)$ be in $C^3([0,l^i])$. Then it holds*

$$\begin{aligned}
&\gamma^1\{v^1_{\sigma\sigma}+(\kappa_*^1)^2v^1\}+\gamma^2\{v^2_{\sigma\sigma}+(\kappa_*^2)^2v^2\}+\gamma^3\{v^3_{\sigma\sigma}+(\kappa_*^3)^2v^3\}=0,\\
&m^1\gamma^1\{v^1_{\sigma\sigma}+(\kappa_*^1)^2v^1\}_\sigma=m^2\gamma^2\{v^2_{\sigma\sigma}+(\kappa_*^2)^2v^2\}_\sigma=m^3\gamma^3\{v^3_{\sigma\sigma}+(\kappa_*^3)^2v^3\}_\sigma
\end{aligned}$$

at $\sigma=0$ and

$$v^i_\sigma+h^i_*v^i=0,\quad \{v^i_{\sigma\sigma}+(\kappa_*^i)^2v^i\}_\sigma=0$$

at $\sigma=l^i$. Here the h^i_* $(i=1,2,3)$ are the curvatures of $\partial\Omega$ at $\Gamma^i_*\cap\partial\Omega$, where we use the sign convention that $h^i_*<0$ $(i=1,2,3)$ if Ω is convex.

Using Lemma 3.1, the proof of this lemma is similar to [7, the proof of Lemma 3.2 and Lemma 3.3]. Thus we omit it.

By means of Lemma 3.2, Leamma 3.3(i), Lemma 3.4, and Lemma 3.5, we obtain the linearized problem for $t>0$

$$v^i_t=-m^i\gamma^i\{v^i_{\sigma\sigma}+(\kappa_*^i)^2v^i\}_{\sigma\sigma},\quad \sigma\in(0,l^i) \tag{3.5}$$

with the boundary conditions

$$\begin{cases}
\gamma^1v^1+\gamma^2v^2+\gamma^3v^3=0,\\
\dfrac{1}{s^1}(c^2\kappa_*^2-c^3\kappa_*^3)v^1+v^1_\sigma=\dfrac{1}{s^2}(c^3\kappa_*^3-c^1\kappa_*^1)v^2+v^2_\sigma=\dfrac{1}{s^3}(c^1\kappa_*^1-c^2\kappa_*^2)v^3+v^3_\sigma,\\
\gamma^1\{v^1_{\sigma\sigma}+(\kappa_*^1)^2v^1\}+\gamma^2\{v^2_{\sigma\sigma}+(\kappa_*^2)^2v^2\}+\gamma^3\{v^3_{\sigma\sigma}+(\kappa_*^3)^2v^3\}=0,\\
m^1\gamma^1\{v^1_{\sigma\sigma}+(\kappa_*^1)^2v^1\}_\sigma=m^2\gamma^2\{v^2_{\sigma\sigma}+(\kappa_*^2)^2v^2\}_\sigma=m^3\gamma^3\{v^3_{\sigma\sigma}+(\kappa_*^3)^2v^3\}_\sigma
\end{cases} \tag{3.6}$$

at $\sigma=0$ and

$$\begin{cases}
v^i_\sigma+h^i_*v^i=0,\\
\{v^i_{\sigma\sigma}+(\kappa_*^i)^2v^i\}_\sigma=0
\end{cases} \tag{3.7}$$

at $\sigma=l^i$ $(i=1,2,3)$ where $c^i:=\cos\theta^i$ and $s^i:=\sin\theta^i$.

4 Gradient flow structure

Let us consider the gradient flow structure of the linearized problem (3.5)-(3.7). For $k \in \mathbb{N}$, set

$$
\begin{aligned}
&\mathcal{H}^k := H^k(0,l^1) \times H^k(0,l^2) \times H^k(0,l^3),\\
&(\mathcal{H}^k)' := (H^k(0,l^1))' \times (H^k(0,l^2))' \times (H^k(0,l^3))',\\
&\mathcal{Y} := \Big\{(\xi^1,\xi^2,\xi^3) \in \mathcal{H}^1 \,\Big|\, \xi^1+\xi^2+\xi^3 = 0 \ \text{ at } \ \sigma=0,\\
&\qquad\qquad \int_0^{l^1} \xi^1\,d\sigma = \int_0^{l^2} \xi^2\,d\sigma = \int_0^{l^3} \xi^3\,d\sigma\Big\},\\
&\mathcal{E} := \Big\{(v^1,v^2,v^3) \in \mathcal{H}^1 \,\Big|\, \gamma^1 v^1+\gamma^2 v^2+\gamma^3 v^3 = 0 \ \text{ at } \ \sigma=0,\\
&\qquad\qquad \int_0^{l^1} v^1\,d\sigma = \int_0^{l^2} v^2\,d\sigma = \int_0^{l^3} v^3\,d\sigma\Big\},\\
&\mathcal{X} := \Big\{(w^1,w^2,w^3) \in (\mathcal{H}^1)' \,|\, \langle w^1,1\rangle = \langle w^2,1\rangle = \langle w^3,1\rangle\Big\},
\end{aligned}
$$

where $H^k(0,l^i)$ is the classical Sobolev space, $(H^k(0,l^i))'$ is the duality space of $H^k(0,l^i)$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between $(H^1(0,l^i))'$ and $H^1(0,l^i)$.

Definition 4.1 *We say that $\boldsymbol{u_w} = (u^1_{\boldsymbol{w}}, u^2_{\boldsymbol{w}}, u^3_{\boldsymbol{w}}) \in \mathcal{Y}$ for given $\boldsymbol{w} = (w^1,w^2,w^3) \in \mathcal{X}$ is a weak solution of*

$$
\begin{cases}
-m^i\partial_\sigma^2 u^i_{\boldsymbol{w}} = w^i & \textit{for} \quad \sigma\in(0,l^i) \quad (i=1,2,3),\\
u^1_{\boldsymbol{w}}+u^2_{\boldsymbol{w}}+u^3_{\boldsymbol{w}} = 0 & \textit{at} \quad \sigma=0,\\
m^1\partial_\sigma u^1_{\boldsymbol{w}} = m^2\partial_\sigma u^2_{\boldsymbol{w}} = m^3\partial_\sigma u^3_{\boldsymbol{w}} & \textit{at} \quad \sigma=0,\\
\partial_\sigma u^i_{\boldsymbol{w}} = 0 & \textit{at} \quad \sigma=l^i \quad (i=1,2,3),
\end{cases}
\tag{4.1}
$$

if $\boldsymbol{u_w} = (u^1_{\boldsymbol{w}}, u^2_{\boldsymbol{w}}, u^3_{\boldsymbol{w}}) \in \mathcal{Y}$ satisfies

$$
\langle \boldsymbol{w},\boldsymbol{\xi}\rangle = \sum_{i=1}^3 m^i\int_0^{l^i} \partial_\sigma u^i_{\boldsymbol{w}}\partial_\sigma\xi^i\,d\sigma \tag{4.2}
$$

for all $\boldsymbol{\xi} = (\xi^1,\xi^2,\xi^3) \in \mathcal{Y}$.

Remark 4.2 Assume that $\boldsymbol{u_w}$ is a smooth solution of (4.1). Then the duality pairing between $\boldsymbol{w} \in (\mathcal{H}^1)'$ and $\boldsymbol{\xi} \in \mathcal{H}^1$ reduces to

$$
\langle \boldsymbol{w},\boldsymbol{\xi}\rangle = \sum_{i=1}^3 m^i\int_0^{l^i} (-\partial_\sigma^2 u^i_{\boldsymbol{w}})\,\xi^i\,d\sigma.
$$

It follows from the integration by parts formula that

$$
\langle \boldsymbol{w},\boldsymbol{\xi}\rangle = \sum_{i=1}^3 m^i\left\{\left[(-\partial_\sigma u^i_{\boldsymbol{w}})\,\xi^i\right]_{\sigma=0}^{\sigma=l^i} + \int_0^{l^i} \partial_\sigma u^i_{\boldsymbol{w}}\partial_\sigma\xi^i\,d\sigma\right\}.
$$

Using in addition the condition $\xi^1+\xi^2+\xi^3=0$ at $\sigma=0$ for $\boldsymbol{\xi}\in\mathcal{H}^1$, the boundary terms vanish by means of $m^1\partial_\sigma u^1_{\boldsymbol{w}} = m^2\partial_\sigma u^2_{\boldsymbol{w}} = m^3\partial_\sigma u^3_{\boldsymbol{w}}$ at $\sigma=0$ and $\partial_\sigma u^i_{\boldsymbol{w}}=0$ at $\sigma=l^i$ ($i=1,2,3$). Hence $\boldsymbol{u_w}$ satisfies (4.2) for all $\boldsymbol{\xi}\in\mathcal{Y}\subset\widetilde{\mathcal{Y}} := \{\boldsymbol{\xi}\in\mathcal{H}^1 \,|\, \xi^1+\xi^2+\xi^3=0 \text{ at } \sigma=0\}$.

Conversely, let $\boldsymbol{u}_{\boldsymbol{w}}$ with $u^1_{\boldsymbol{w}} + u^2_{\boldsymbol{w}} + u^3_{\boldsymbol{w}} = 0$ at $\sigma = 0$ be a smooth solution of the weak formulation (4.2). We first show that (4.2) holds for all $\widetilde{\boldsymbol{\xi}} \in \widetilde{\mathcal{Y}}$. For each $\widetilde{\boldsymbol{\xi}} \in \widetilde{\mathcal{Y}}$, there exist constants (c^1, c^2, c^3) such that

$$(\widetilde{\xi}^1 - c^1, \widetilde{\xi}^2 - c^2, \widetilde{\xi}^3 - c^3) \in \mathcal{Y}.$$

Indeed, taking into account $\widetilde{\xi}^1 + \widetilde{\xi}^2 + \widetilde{\xi}^3 = 0$ at $\sigma = 0$, we need to find (c^1, c^2, c^3) satisfying

$$c^1 + c^2 + c^3 = 0, \quad \int_0^{l^1} \widetilde{\xi}^1 \, d\sigma - c^1 l^1 = \int_0^{l^2} \widetilde{\xi}^2 \, d\sigma - c^2 l^2 = \int_0^{l^3} \widetilde{\xi}^3 \, d\sigma - c^3 l^3. \tag{4.3}$$

Since the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ -l^1 & l^2 & 0 \\ 0 & -l^2 & l^3 \end{pmatrix}$$

is invertible, there exist constants (c^1, c^2, c^3) such that (4.3) is fulfilled. Thus, using $(\widetilde{\xi}^1 - c^1, \widetilde{\xi}^2 - c^2, \widetilde{\xi}^3 - c^3) \in \mathcal{Y}$ as a test function in (4.2), it follows from $\langle w^1, 1\rangle = \langle w^2, 1\rangle = \langle w^3, 1\rangle$ and $c^1 + c^2 + c^3 = 0$ that

$$\sum_{i=1}^{3} \langle w^i, \widetilde{\xi}^i - c^i\rangle = \sum_{i=1}^{3} \langle w^i, \widetilde{\xi}^i\rangle - \sum_{i=1}^{3} c^i \langle w^i, 1\rangle = \sum_{i=1}^{3} \langle w^i, \widetilde{\xi}^i\rangle.$$

This implies that (4.2) holds for all $\widetilde{\boldsymbol{\xi}} \in \widetilde{\mathcal{Y}}$. The integration by parts formula gives

$$\langle \boldsymbol{w}, \widetilde{\boldsymbol{\xi}}\rangle = \sum_{i=1}^{3} m^i \left\{ \left[(\partial_\sigma u^i_{\boldsymbol{w}})\, \widetilde{\xi}^i \right]_{\sigma=0}^{\sigma=l^i} + \int_0^{l^i} (-\partial_\sigma^2 u^i_{\boldsymbol{w}})\, \widetilde{\xi}^i \, d\sigma \right\}$$

for all $\widetilde{\boldsymbol{\xi}} \in \widetilde{\mathcal{Y}}$. Then the equation $-m^i \partial_\sigma^2 u^i_{\boldsymbol{w}} = w^i$ holds pointwise. Further, we have $\sum_{i=1}^{3} m^i \left[(\partial_\sigma u^i_{\boldsymbol{w}})\, \widetilde{\xi}^i \right]_{\sigma=0}^{\sigma=l^i} = 0$. Since $\widetilde{\boldsymbol{\xi}} \in \mathcal{H}^1$ is arbitrary as long as $\widetilde{\xi}^1 + \widetilde{\xi}^2 + \widetilde{\xi}^3 = 0$ at $\sigma = 0$, we obtain $m^1 \partial_\sigma u^1_{\boldsymbol{w}} = m^2 \partial_\sigma u^2_{\boldsymbol{w}} = m^3 \partial_\sigma u^3_{\boldsymbol{w}}$ at $\sigma = 0$ and $\partial_\sigma u^i_{\boldsymbol{w}} = 0$ at $\sigma = l^i$ $(i = 1, 2, 3)$. Hence $\boldsymbol{u}_{\boldsymbol{w}}$ is a solution of (4.1).

Remark 4.3 For each $\boldsymbol{w} \in \mathcal{X}$, there exists a unique weak solution of (4.1) in $\mathcal{Y}$. Indeed, set

$$\mathcal{I}[\boldsymbol{u}, \boldsymbol{\xi}] := \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma u^i \partial_\sigma \xi^i \, d\sigma \tag{4.4}$$

for all $\boldsymbol{u}, \boldsymbol{\xi} \in \mathcal{Y}$. Since a Poincaré-type inequality holds for all $\boldsymbol{u} \in \mathcal{Y}$ (cf. Lemma 5.2 in Section 5), the bilinear form $\mathcal{I}$ is continuous and coercive on $\mathcal{Y}$. Then it follows from the Lax-Milgram theorem that for each $\boldsymbol{w} \in \mathcal{X}$ there exists a unique $\boldsymbol{u}_{\boldsymbol{w}} \in \mathcal{Y}$ such that

$$\mathcal{I}[\boldsymbol{u}_{\boldsymbol{w}}, \boldsymbol{\xi}] = \langle \boldsymbol{w}, \boldsymbol{\xi}\rangle$$

for all $\boldsymbol{\xi} \in \mathcal{Y}$. This shows the above assertion.

Definition 4.4 *For a given $\boldsymbol{w} = (w^1, w^2, w^3) \in \mathcal{X}$, we say that $\boldsymbol{v} = (v^1, v^2, v^3) \in \mathcal{H}^3$ with*

$$\int_0^{l^1} v^1 \, d\sigma = \int_0^{l^2} v^2 \, d\sigma = \int_0^{l^3} v^3 \, d\sigma$$

is a weak solution of

$$w^i = -m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma} \quad \text{for} \;\; \sigma \in (0, l^i) \tag{4.5}$$

with the boundary conditions (3.6) at $\sigma = 0$ *and (3.7) at* $\sigma = l^i$ *if* $\boldsymbol{v}$ *satisfies*

$$\langle \boldsymbol{w}, \boldsymbol{\xi}\rangle = \sum_{i=1}^{3} m^i \int_0^{l^i} \gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_\sigma \xi^i_\sigma \, d\sigma$$

for all $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3) \in \mathcal{Y}$ *and fulfills the boundary conditions*

$$\begin{cases} \gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0, \\ \dfrac{1}{s^1}(c^2\kappa^2_* - c^3\kappa^3_*)v^1 + v^1_\sigma = \dfrac{1}{s^2}(c^3\kappa^3_* - c^1\kappa^1_*)v^2 + v^2_\sigma = \dfrac{1}{s^3}(c^1\kappa^1_* - c^2\kappa^2_*)v^3 + v^3_\sigma, \\ \gamma^1\{v^1_{\sigma\sigma} + (\kappa^1_*)^2 v^1\} + \gamma^2\{v^2_{\sigma\sigma} + (\kappa^2_*)^2 v^2\} + \gamma^3\{v^3_{\sigma\sigma} + (\kappa^3_*)^2 v^3\} = 0 \end{cases} \tag{4.6}$$

at $\sigma = 0$ *and*

$$v^i_\sigma + h^i_* v^i = 0 \tag{4.7}$$

at $\sigma = l^i$ *(*$i = 1, 2, 3$*).*

Define the symmetric bilinear form

$$\begin{aligned} I[\boldsymbol{v}_1, \boldsymbol{v}_2] := & \sum_{i=1}^{3} \gamma^i \int_0^{l^i} \{v^i_{1,\sigma} v^i_{2,\sigma} - (\kappa^i_*)^2 v^i_1 v^i_2\} d\sigma + \sum_{i=1}^{3} \gamma^i h^i_* v^i_1 v^i_2 \Big|_{\sigma = l^i} \\ & - \frac{\gamma^1}{s^1}(c^2\kappa^2_* - c^3\kappa^3_*) v^1_1 v^1_2 \Big|_{\sigma=0} - \frac{\gamma^2}{s^2}(c^3\kappa^3_* - c^1\kappa^1_*) v^2_1 v^2_2 \Big|_{\sigma=0} \\ & - \frac{\gamma^3}{s^3}(c^1\kappa^1_* - c^2\kappa^2_*) v^3_1 v^3_2 \Big|_{\sigma=0} \end{aligned}$$

and the inner product

$$(\boldsymbol{v}_1, \boldsymbol{v}_2)_{-1} := \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma u^i_{\boldsymbol{v}_1} \partial_\sigma u^i_{\boldsymbol{v}_2} \, d\sigma, \tag{4.8}$$

where $\boldsymbol{u}_{\boldsymbol{v}_j} = (u^1_{\boldsymbol{v}_j}, u^2_{\boldsymbol{v}_j}, u^3_{\boldsymbol{v}_j})$ ($j = 1, 2$) are weak solutions of (4.1) for given $\boldsymbol{v}_j = (v^1_j, v^2_j, v^3_j) \in \mathcal{X}$.

Lemma 4.5 *Let* $\boldsymbol{w} = (w^1, w^2, w^3) \in \mathcal{X}$ *and* $\boldsymbol{v} = (v^1, v^2, v^3) \in \mathcal{H}^1$ *be given. Then the following two states are equivalent.*

(i) $\boldsymbol{v} \in \mathcal{H}^3$ *with*

$$\int_0^{l^1} v^1 \, d\sigma = \int_0^{l^2} v^2 \, d\sigma = \int_0^{l^3} v^3 \, d\sigma$$

and $\boldsymbol{v}$ *is a weak solution of (4.5) with the boundary conditions (3.6) at* $\sigma = 0$ *and (3.7) at* $\sigma = l^i$*.*

(ii) $\boldsymbol{v} \in \mathcal{E}$ *and* $\boldsymbol{v}$ *fulfills*

$$(\boldsymbol{w}, \boldsymbol{\varphi})_{-1} = -I[\boldsymbol{v}, \boldsymbol{\varphi}] \tag{4.9}$$

for all $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3) \in \mathcal{E}$*.*

Proof. Assume that (i) holds. Then $\boldsymbol{v} \in \mathcal{H}^3$ is a weak solution of the linearized system (4.5), (3.6), (3.7). Set

$$\Xi^i := \gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\} \quad (i = 1, 2, 3).$$

Note that $(\Xi^1, \Xi^2, \Xi^3) \in \mathcal{H}^1$ satisfies $\Xi^1 + \Xi^2 + \Xi^3 = 0$ at $\sigma = 0$. By (4.8), Definition 4.1, and Definition 4.4, we see

$$(\boldsymbol{w}, \boldsymbol{\varphi})_{-1} = \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma u^i_{\boldsymbol{w}} \partial_\sigma u^i_{\boldsymbol{\varphi}}\, d\sigma = \langle \boldsymbol{w}, \boldsymbol{u}_{\boldsymbol{\varphi}} \rangle = \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma \Xi^i \partial_\sigma u^i_{\boldsymbol{\varphi}}\, d\sigma$$

for all $\boldsymbol{\varphi} \in \mathcal{E}$, where $\boldsymbol{u}_{\boldsymbol{\varphi}} = (u^1_{\boldsymbol{\varphi}}, u^2_{\boldsymbol{\varphi}}, u^3_{\boldsymbol{\varphi}}) \in \mathcal{Y}$. According to Remark 4.2, we can find (c^1, c^2, c^3) such that $(\Xi^1 - c^1, \Xi^2 - c^2, \Xi^3 - c^3) \in \mathcal{Y}$. That is, the c^i $(i = 1, 2, 3)$ satisfy

$$c^1 + c^2 + c^3 = 0, \qquad \int_0^{l^1} \Xi^1\, d\sigma - c^1 l^1 = \int_0^{l^2} \Xi^2\, d\sigma - c^2 l^2 = \int_0^{l^3} \Xi^3\, d\sigma - c^3 l^3. \tag{4.10}$$

Then, using the fact that $\boldsymbol{u}_{\boldsymbol{\varphi}} \in \mathcal{Y}$ is a weak solution of (4.1), we obtain

$$\sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma \Xi^i \partial_\sigma u^i_{\boldsymbol{\varphi}}\, d\sigma = \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma (\Xi^i - c^i) \partial_\sigma u^i_{\boldsymbol{\varphi}}\, d\sigma = \sum_{i=1}^{3} \int_0^{l^i} (\Xi^i - c^i) \varphi^i\, d\sigma.$$

Further, it follows from $\boldsymbol{\varphi} \in \mathcal{E}$ and $c^1 + c^2 + c^3 = 0$ that

$$\sum_{i=1}^{3} \int_0^{l^i} (\Xi^i - c^i) \varphi^i\, d\sigma = \sum_{i=1}^{3} \int_0^{l^i} \Xi^i \varphi^i\, d\sigma.$$

Thus we have

$$\begin{aligned}
(\boldsymbol{w}, \boldsymbol{\varphi})_{-1} &= \sum_{i=1}^{3} \int_0^{l^i} \gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\} \varphi^i\, d\sigma \\
&= \sum_{i=1}^{3} \gamma^i \int_0^{l^i} v^i_{\sigma\sigma} \varphi^i\, d\sigma + \sum_{i=1}^{3} \gamma^i (\kappa^i_*)^2 \int_0^{l^i} v^i \varphi^i\, d\sigma \\
&= \sum_{i=1}^{3} \gamma^i \left[v^i_\sigma \varphi^i \right]_{\sigma=0}^{\sigma=l^i} - \sum_{i=1}^{3} \gamma^i \int_0^{l^i} v^i_\sigma \varphi^i_\sigma + \sum_{i=1}^{3} \gamma^i (\kappa^i_*)^2 \int_0^{l^i} v^i \varphi^i\, d\sigma.
\end{aligned}$$

Using $\gamma^1\varphi^1 + \gamma^2\varphi^2 + \gamma^3\varphi^3 = 0$ at $\sigma = 0$, the second condition in (3.6), and the first condition in (3.7), we are led to (4.9).

Conversely, assume that (ii) holds. Then $\boldsymbol{v} \in \mathcal{E}$ fulfills (4.9) for all $\boldsymbol{\varphi} \in \mathcal{E}$. For each $\boldsymbol{\varphi} \in \mathcal{E}$, there exists $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \zeta^3) \in \mathcal{H}^3$ such that

$$\begin{cases}
-m^i \zeta^i_{\sigma\sigma} = \varphi^i & \text{for} \quad \sigma \in (0, l^i), \\
\zeta^1 + \zeta^2 + \zeta^3 = 0 & \text{at} \quad \sigma = 0, \\
m^1 \zeta^1_\sigma = m^2 \zeta^2_\sigma = m^3 \zeta^3_\sigma & \text{at} \quad \sigma = 0, \\
\zeta^i_\sigma = 0 & \text{at} \quad \sigma = l^i.
\end{cases}$$

In fact, to obtain such a $\boldsymbol{\zeta}$, we consider the minimizing problem

$$\frac{1}{2}\mathcal{I}[\boldsymbol{\zeta},\boldsymbol{\zeta}] - \sum_{i=1}^{3}\int_0^{l^i} \varphi^i\zeta^i\, d\sigma \;\to\; \min$$

for all $\boldsymbol{\zeta} \in \mathcal{Y}$, where $\mathcal{I}$ is the bilinear form defined as (4.4). Since a Poincaré-type inequality holds for all $\boldsymbol{\zeta} \in \mathcal{Y}$ and $\mathcal{I}$ is symmetric, this minimizing problem admits a solution $\boldsymbol{\zeta} \in \mathcal{Y}$. Further, applying the regularity theory, we get a $\boldsymbol{\zeta}$ with the required regularity. Then, using this $\boldsymbol{\zeta}$ and taking into account $(\boldsymbol{w},\boldsymbol{\varphi})_{-1} = \langle \boldsymbol{w},\boldsymbol{\zeta}\rangle$, we obtain

$$\begin{aligned}
\langle \boldsymbol{w},\boldsymbol{\zeta}\rangle = {} & \sum_{i=1}^{3} m^i\gamma^i \int_0^{l^i} \{v^i_\sigma \zeta^i_{\sigma\sigma\sigma} - (\kappa^i_*)^2 v^i\zeta^i_{\sigma\sigma}\}d\sigma + \sum_{i=1}^{3} m^i\gamma^i h^i_* v^i\zeta^i_{\sigma\sigma}\Big|_{\sigma=l^i} \\
& - \frac{m^1\gamma^1}{s^1}(c^2\kappa^2_* - c^3\kappa^3_*)v^1\zeta^1_{\sigma\sigma}\Big|_{\sigma=0} - \frac{m^2\gamma^2}{s^2}(c^3\kappa^3_* - c^1\kappa^1_*)v^2\zeta^2_{\sigma\sigma}\Big|_{\sigma=0} \\
& - \frac{m^3\gamma^3}{s^3}(c^1\kappa^1_* - c^2\kappa^2_*)v^3\zeta^3_{\sigma\sigma}\Big|_{\sigma=0}.
\end{aligned}$$

It follows from $\boldsymbol{w} \in (\mathcal{H}^1)'$ that $\boldsymbol{v} \in \mathcal{H}^3$. Further, the integration by parts formula gives

$$\begin{aligned}
\langle \boldsymbol{w},\boldsymbol{\zeta}\rangle = {} & \sum_{i=1}^{3} m^i\gamma^i \int_0^{l^i} \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_\sigma \zeta^i_\sigma\, d\sigma + \sum_{i=1}^{3} m^i\gamma^i \{v^i_\sigma + h^i_* v^i\}\zeta^i_{\sigma\sigma}\Big|_{\sigma=l^i} \\
& - m^1\gamma^1\Big\{v^1_\sigma + \frac{1}{s^1}(c^2\kappa^2_* - c^3\kappa^3_*)v^1\Big\}\zeta^1_{\sigma\sigma}\Big|_{\sigma=0} \\
& - m^2\gamma^2\Big\{v^2_\sigma + \frac{1}{s^2}(c^3\kappa^3_* - c^1\kappa^1_*)v^2\Big\}\zeta^2_{\sigma\sigma}\Big|_{\sigma=0} \\
& - m^3\gamma^3\Big\{v^3_\sigma + \frac{1}{s^3}(c^1\kappa^1_* - c^2\kappa^2_*)v^3\Big\}\zeta^3_{\sigma\sigma}\Big|_{\sigma=0} \\
& + \sum_{i=1}^{3} m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}\zeta^i_\sigma|_{\sigma=0}.
\end{aligned} \tag{4.11}$$

We first observe that the parts which except the boundary terms from (4.11) are a weak formulation of $w^i = -m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma}$ for $\sigma \in (0, l^i)$ together with

$$\begin{cases} m^1\{v^1_{\sigma\sigma} + (\kappa^1_*)^2 v^1\}_\sigma = m^2\{v^2_{\sigma\sigma} + (\kappa^2_*)^2 v^2\}_\sigma = m^3\{v^3_{\sigma\sigma} + (\kappa^3_*)^2 v^3\}_\sigma \quad \text{at} \quad \sigma = 0, \\ \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_\sigma = 0 \quad \text{at} \quad \sigma = l^i. \end{cases}$$

From the boundary terms of (4.11) at $\sigma = l^i$, we derive (4.7) by using the fact that $\zeta^i_{\sigma\sigma}$ is arbitrarily at $\sigma = l^i$. Note that

$$m^1\gamma^1\zeta^1_{\sigma\sigma} + m^2\gamma^2\zeta^2_{\sigma\sigma} + m^3\gamma^3\zeta^3_{\sigma\sigma} = 0 \quad \text{at} \quad \sigma = 0. \tag{4.12}$$

Using $m^1\zeta^1_\sigma = m^2\zeta^2_\sigma = m^3\zeta^3_\sigma$ at $\sigma = 0$ and (4.12) in the boundary terms of (4.11) at $\sigma = 0$, we obtain the second and third conditions in (4.6). Thus $\boldsymbol{v}$ is a weak solution of the linearized system (4.5), (3.6), (3.7). $\square$

5 Self-adjointness of the linearized operator

Set

$$\mathcal{D}(\mathcal{A}) = \{\boldsymbol{v} = (v^1, v^2, v^3) \in \mathcal{H}^3 \,|\, \boldsymbol{v} \text{ satisfies (4.6) at } \sigma = 0, \text{ (4.7) at } \sigma = l^i, \text{ and}$$
$$\int_0^{l^1} v^1 \, d\sigma = \int_0^{l^2} v^2 \, d\sigma = \int_0^{l^3} v^3 \, d\sigma\}.$$

Then we define the linearized operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ as

$$\langle \mathcal{A}\boldsymbol{v}, \boldsymbol{\xi} \rangle = \sum_{i=1}^{3} m^i \int_0^{l^i} \gamma^i \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_\sigma \xi^i_\sigma \, d\sigma$$

for all $\boldsymbol{v} \in \mathcal{D}(\mathcal{A})$ and $\boldsymbol{\xi} \in \mathcal{Y}$. This definition gives for all $\boldsymbol{\varphi} \in \mathcal{E}$

$$(\mathcal{A}\boldsymbol{v}, \boldsymbol{\varphi})_{-1} = -I[\boldsymbol{v}, \boldsymbol{\varphi}].$$

For this operator $\mathcal{A}$, we have the following lemma.

Lemma 5.1 *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(\cdot, \cdot)_{-1}$.*

Proof. Since $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{D}(\mathcal{A})$ implies $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{E}$, we have

$$(\mathcal{A}\boldsymbol{v}_1, \boldsymbol{v}_2)_{-1} = -I[\boldsymbol{v}_1, \boldsymbol{v}_2] = -I[\boldsymbol{v}_2, \boldsymbol{v}_1] = (\mathcal{A}\boldsymbol{v}_2, \boldsymbol{v}_1)_{-1} = (\boldsymbol{v}_2, \mathcal{A}\boldsymbol{v}_1)_{-1}.$$

Thus $\mathcal{A}$ is symmetric. □

Now we use the notation

$$\|\boldsymbol{v}\| = \left(\sum_{i=1}^{3} \|v^i\|^2 \right)^{1/2}, \quad \|\boldsymbol{v}\|_{-1} = \left(\sum_{i=1}^{3} \|v^i\|^2_{-1} \right)^{1/2}$$

for $\boldsymbol{v} = (v^1, v^2, v^3)$, where $\|\cdot\|$ is the L^2-norm and $\|\cdot\|_{-1} = \{(\cdot, \cdot)_{-1}\}^{1/2}$. The following lemmas will be needed.

Lemma 5.2 *(Poincaré-type inequality) For all $\boldsymbol{v} \in \mathcal{E}$, there exists a $C > 0$, which depends on l^i $(i = 1, 2, 3)$, such that*

$$\|\boldsymbol{v}\| \leq C\|\boldsymbol{v}_\sigma\|.$$

Proof. First we prove that there are $j \in \{1, 2, 3\}$ and $\sigma_j \in [0, l^j]$ such that $v^j(\sigma_j) = 0$. We show the assertion by a contradiction. If not, then v^i has a definite sign for each $i \in \{1, 2, 3\}$ and $\sigma \in [0, l^i]$. Since $\boldsymbol{v} \in \mathcal{E}$, v^i $(i = 1, 2, 3)$ satisfy

$$\int_0^{l^1} v^1 \, d\sigma = \int_0^{l^2} v^2 \, d\sigma = \int_0^{l^3} v^3 \, d\sigma. \tag{5.1}$$

This implies that v^i must have the same sign for $i = 1, 2, 3$ and $\sigma \in [0, l^i]$. But this contradicts to $\gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0$ at $\sigma = 0$.

By the above fact, we derive

$$|v^j(\sigma)| = \Big|\int_{\sigma_j}^{\sigma} v_r^j \, dr\Big| \leq \int_0^{l^j} |v_\sigma^j| \, d\sigma \leq (l^j)^{1/2} \|v_\sigma^j\|.$$

Then it follows from (5.1) that

$$\Big|\int_0^{l^i} v^i d\sigma\Big| = \Big|\int_0^{l^j} v^j d\sigma\Big| \leq \int_0^{l^j} |v^j| d\sigma \leq (l^j)^{3/2} \|v_\sigma^j\| \leq (l^j)^{3/2} \|\boldsymbol{v}_\sigma\|. \tag{5.2}$$

Setting $v_{av}^i = (l^i)^{-1} \displaystyle\int_0^{l^i} v^i \, d\sigma$, (5.2) implies

$$\|v_{av}^i\| \leq (l^i)^{-1/2} (l^j)^{3/2} \|\boldsymbol{v}_\sigma\|. \tag{5.3}$$

By means of $\displaystyle\int_0^{l^i} (v^i - v_{av}^i) \, d\sigma = 0$ and Schwarz's inequality, we obtain

$$|v^i - v_{av}^i| \leq \int_0^{l^i} |v_\sigma^i| \, d\sigma \leq (l^i)^{1/2} \|v_\sigma^i\|,$$

so that

$$\|v^i - v_{av}^i\| \leq l^i \|v_\sigma^i\| \leq l^i \|\boldsymbol{v}_\sigma\|. \tag{5.4}$$

By (5.3) and (5.4), we are led to the desired inequality. $\square$

Lemma 5.3 *For all $\delta > 0$ there exists a constant $C_\delta > 0$ such that for all $\boldsymbol{v} \in \mathcal{E}$ and each $i = 1, 2, 3$ the inequality*

$$|v^i(0)|^2 \leq \delta \|\boldsymbol{v}_\sigma\|^2 + C_\delta \|\boldsymbol{v}\|_{-1}^2$$

holds. The same inequality holds for $v^i(l^i)$ instead of $v^i(0)$.

Applying a similar argument to the proof of [7, Lemma 5.2] and using Lemma 5.2, we can prove this lemma. Thus we omit the proof.

Lemma 5.4 *There exist $c_1 > 0$ and $c_2 > 0$ such that*

$$\|\boldsymbol{v}_\sigma\|^2 \leq c_1 \|\boldsymbol{v}\|_{-1}^2 + c_2 I[\boldsymbol{v}, \boldsymbol{v}] \quad \textit{for all } \boldsymbol{v} \in \mathcal{E}.$$

The proof of this lemma is similar to that of [7, Lemma 5.3]. Thus we omit it.

Also, applying a similar argument as in the proof of [7, Corollary 5.4], the Lemma 5.4 implies the following corollary.

Corollary 5.5 *The largest eigenvalue of $\mathcal{A}$ is bounded from above by c_1/c_2, where c_1, c_2 are as in Lemma 5.4.*

Then it follows from this corollary that

$$(\mathcal{A}\boldsymbol{v}, \boldsymbol{v})_{-1} \leq c \|\boldsymbol{v}\|_{-1} \tag{5.5}$$

with $c = c_1/c_2$.

By means of the above lemmas, we obtain the following theorem.

Theorem 5.6 *(i) The operator $\mathcal{A}$ is self-adjoint with respect to the inner product $(\cdot,\cdot)_{-1}$.*

(ii) The spectrum of $\mathcal{A}$ contains a countable system of eigenvalues.

(iii) The initial value problem (3.5)-(3.7) is solvable for given initial data in $\mathcal{X}$.

(iv) The zero solution is an asymptotically stable solution of (3.5)-(3.7) if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Proof. Let us prove (i). By means of (5.5), we find a $\omega \in \mathbb{R}$ such that the operator $\mathcal{B} := \omega Id - \mathcal{A}$ is strongly monotone with respect to the inner product $(\cdot,\cdot)_{-1}$. Since $\mathcal{A}$ is symmetric (see Lemma 5.1), $\mathcal{B}$ is also symmetric. We show that $\mathcal{R}(\mathcal{B}) = \mathcal{X}$, where $\mathcal{R}(\mathcal{B})$ is the range of the operator $\mathcal{B}$. For each $\boldsymbol{f} = (f^1, f^2, f^3) \in \mathcal{X}$, we need to prove that there exists a weak solution $\boldsymbol{v}$ of the boundary value problem

$$\begin{cases} m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma} + \omega v^i = f^i \ \text{ for } \ \sigma \in (0, l^i) \ \ (i = 1,2,3), \\ (3.6) \text{ at } \sigma = 0 \text{ and } (3.7) \text{ at } \sigma = l^i \ (i = 1,2,3). \end{cases} \tag{5.6}$$

To obtain a weak solution of (5.6), we consider the minimizing problem

$$F(\boldsymbol{v}) := \frac{1}{2}\{I[\boldsymbol{v},\boldsymbol{v}] + \omega\|\boldsymbol{v}\|^2_{-1}\} - (\boldsymbol{f},\boldsymbol{v})_{-1} \ \rightarrow \ \min$$

for all $\boldsymbol{v} \in \mathcal{E}$. It follows from Lemma 5.4 that F is coercive on $\mathcal{E}$ for sufficiently large ω, so that this minimizing problem admits a solution $\widetilde{\boldsymbol{v}} = (\widetilde{v}^1, \widetilde{v}^2, \widetilde{v}^3)$ when ω is large enough. Taking the first variation of F, we have

$$-\gamma^i\{\widetilde{v}^i_{\sigma\sigma} + (\kappa^i_*)^2\widetilde{v}^i\} + c^i + \omega u^i_{\widetilde{\boldsymbol{v}}} = u^i_{\boldsymbol{f}} \ \text{ for } \ \sigma \in (0, l^i) \ \ (i = 1,2,3) \tag{5.7}$$

with the first and second conditions in (3.6) at $\sigma = 0$ and the first condition in (3.7) at $\sigma = l^i \, (i = 1,2,3)$, where $c^i \, (i = 1,2,3)$ are constants satisfying (4.10). Then it follows from $u^i_{\widetilde{\boldsymbol{v}}}, u^i_{\boldsymbol{f}} \in H^1(0, l^i) \, (i = 1,2,3)$ that $\widetilde{\boldsymbol{v}} \in \mathcal{H}^3$. The sum of (5.7) for $i = 1,2,3$ leads us to the third condition in (3.6) at $\sigma = 0$ and differentiating (5.7) twice in a weak sense gives a weak formulation of $m^i\gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma} + \omega v^i = f^i$ together with the remaining boundary conditions in (3.6) and (3.7) (cf. Definition 4.4). Thus, for each $\boldsymbol{f} \in \mathcal{X}$, there exists $\widetilde{\boldsymbol{v}} \in \mathcal{D}(\mathcal{B}) \, (= \mathcal{D}(\mathcal{A}))$ such that $\mathcal{B}\widetilde{\boldsymbol{v}} = \boldsymbol{f}$. That is, we obtain $\mathcal{R}(\mathcal{B}) = \mathcal{X}$. This implies that $\mathcal{B}$ is self-adjoint, so that $\mathcal{A}$ is self-adjoint.

The assertions (ii)-(iv) follow from the standard theory of self-adjoint operators and the theory of semigroups, respectively (see [7, 12, 13]). This competes the proof of the theorem. □

Lemma 5.7 *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

be the eigenvalues of $\mathcal{A}$ (taking into account the multiplicity).

(i) It holds for all $n \in \mathbb{N}$

$$\lambda_n = -\inf_{\mathcal{W}\in\Sigma_{n-1}} \sup_{\boldsymbol{v}\in\mathcal{W}^\perp\setminus\{\boldsymbol{0}\}} \frac{I[\boldsymbol{v},\boldsymbol{v}]}{(\boldsymbol{v},\boldsymbol{v})_{-1}},$$

$$\lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{\boldsymbol{v}\in\mathcal{W}^\perp\setminus\{\boldsymbol{0}\}} \frac{I[\boldsymbol{v},\boldsymbol{v}]}{(\boldsymbol{v},\boldsymbol{v})_{-1}}.$$

Here Σ_n is the collection of n-dimensional subspaces of $\mathcal{E}$ and $\mathcal{W}^\perp$ is the orthogonal complement of $\mathcal{W}$ with respect to the H^{-1}-inner product.

(ii) The eigenvalues depend continuously on h_*^i, l^i, and κ_*^i. Further, the eigenvalues are monotone decreasing in each of the parameters h_*^i $(i = 1, 2, 3)$.

Proof. The lemma follows with the help of Courant's maximum-minimum principle together with the fact that I depends continuously on h_*^i, l^i, and κ_*^i, and is monotone with respect to h_*^i. The proof follows the lines of Courant and Hilbert [1, Chapter VI]. $\square$

6 Stability analysis

In this section, we apply the stability criterion formulated in the previous section to two cases.

6.1 The case: $\kappa_*^i = 0\,(i = 1, 2, 3)$

Lemma 6.1 *(i) The operator $\mathcal{A}$ has zero eigenvalues if and only if (h_*^1, h_*^2, h_*^3) satisfies*

$$\Lambda(h_*^1, h_*^2, h_*^3) = 0$$

with the form (a precise definition is given in the proof)

$$\begin{aligned}\Lambda(h_*^1, h_*^2, h_*^3) := {} & a_{123}h_*^1h_*^2h_*^3 + a_{12}h_*^1h_*^2 + a_{23}h_*^2h_*^3 + a_{31}h_*^3h_*^1 \\ & + a_1h_*^1 + a_2h_*^2 + a_3h_*^3 + a_0,\end{aligned} \tag{6.1}$$

where $a_{123}, a_{12}, \cdots, a_0$ are positive constants depending continuously on γ^i and l^i $(i = 1, 2, 3)$.

(ii) Set

$$\mathcal{S} = \{(h_*^1, h_*^2, h_*^3) \mid \Lambda(h_*^1, h_*^2, h_*^3) = 0\}.$$

The multiplicity of zero eigenvalues is equal to two if $(h_*^1, h_*^2, h_*^3) = (h_{*,c}^1, h_{*,c}^2, h_{*,c}^3) \in \mathcal{S}$, where

$$\begin{cases} h_{*,c}^1 = -\dfrac{a_{31}a_2 + a_{12}a_3 - a_{123}a_0 - a_{23}a_1}{2(a_{31}a_{12} - a_{123}a_1)}, \\ h_{*,c}^2 = -\dfrac{a_{12}a_3 + a_{23}a_1 - a_{123}a_0 - a_{31}a_2}{2(a_{12}a_{23} - a_{123}a_2)}, \\ h_{*,c}^3 = -\dfrac{a_{23}a_1 + a_{31}a_2 - a_{123}a_0 - a_{12}a_3}{2(a_{23}a_{31} - a_{123}a_3)}. \end{cases} \tag{6.2}$$

Further, it is equal to one if $(h_*^1, h_*^2, h_*^3) \in \mathcal{S} \setminus \{(h_{*,c}^1, h_{*,c}^2, h_{*,c}^3)\}$

Proof. In case that $\kappa_*^i = 0\,(i = 1, 2, 3)$, we obtain that eigenfunctions of zero eigenvalues would have the form $v^i(\sigma) := \alpha_3^i\sigma^3 + \alpha_2^i\sigma^2 + \alpha_1^i\sigma + \alpha_0^i$, where α_k^i are constants. Then we have

$$v_\sigma^i(\sigma) = 3\alpha_3^i\sigma^2 + 2\alpha_2^i\sigma + \alpha_1^i, \quad v_{\sigma\sigma}^i(\sigma) = 6\alpha_3^i\sigma + 2\alpha_2^i, \quad v_{\sigma\sigma\sigma}^i(\sigma) = 6\alpha_3^i.$$

The boundary condition $v_{\sigma\sigma\sigma}^i = 0$ at $\sigma = l^i$ gives $\alpha_3^i = 0$. This implies that

$$v^i(\sigma) = \alpha_2^i\sigma^2 + \alpha_1^i\sigma + \alpha_0^i, \quad v_\sigma^i(\sigma) = 2\alpha_2^i\sigma + \alpha_1^i, \quad v_{\sigma\sigma}^i(\sigma) = 2\alpha_2^i.$$

By virtue of the boundary conditions

$$\begin{cases} \gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0, \\ v^1_\sigma = v^2_\sigma = v^3_\sigma, \\ \gamma^1 v^1_{\sigma\sigma} + \gamma^2 v^2_{\sigma\sigma} + \gamma^3 v^3_{\sigma\sigma} = 0, \end{cases}$$

at $\sigma = 0$, we are led to

$$\begin{cases} \gamma^1 \alpha^1_0 + \gamma^2 \alpha^2_0 + \gamma^3 \alpha^3_0 = 0, \\ \alpha^1_1 = \alpha^2_1 = \alpha^3_1, \\ \gamma^1 \alpha^1_2 + \gamma^2 \alpha^2_2 + \gamma^3 \alpha^3_2 = 0. \end{cases} \tag{6.3}$$

Further, by virtue of the boundary condition

$$v^i_\sigma + h^i_* v^i = 0$$

at $\sigma = l^i$, we have

$$(2\alpha^i_2 l^i + \alpha^i_1) + h^i_* \big\{\alpha^i_2 (l^i)^2 + \alpha^i_1 l^i + \alpha^i_0\big\} = 0. \tag{6.4}$$

Since v^i $(i = 1, 2, 3)$ also satisfy

$$\int_0^{l^1} v^1 \, d\sigma = \int_0^{l^2} v^2 \, d\sigma = \int_0^{l^3} v^3 \, d\sigma,$$

we obtain

$$\begin{cases} \dfrac{1}{3}\alpha^1_2 (l^1)^3 + \dfrac{1}{2}\alpha^1_1 (l^1)^2 + \alpha^1_0 l^1 = \dfrac{1}{3}\alpha^2_2 (l^2)^3 + \dfrac{1}{2}\alpha^2_1 (l^2)^2 + \alpha^2_0 l^2 \\ \dfrac{1}{3}\alpha^1_2 (l^1)^3 + \dfrac{1}{2}\alpha^1_1 (l^1)^2 + \alpha^1_0 l^1 = \dfrac{1}{3}\alpha^3_2 (l^3)^3 + \dfrac{1}{2}\alpha^3_1 (l^3)^2 + \alpha^3_0 l^3. \end{cases} \tag{6.5}$$

Then $\lambda = 0$ is an eigenvalue if and only if the equations (6.3)-(6.5) have a nontrivial solution $(\alpha^1_0, \alpha^2_0, \alpha^3_0, \alpha^1_1, \alpha^2_1, \alpha^3_1, \alpha^1_2, \alpha^2_2, \alpha^3_2) \neq 0$, which is equivalent to $\det\left[M(h^1_*, h^2_*, h^3_*)\right] = 0$ where $M(h^1_*, h^2_*, h^3_*)$ is a 9×9-matrix as follows:

$$\begin{bmatrix} \gamma^1 & \gamma^2 & \gamma^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma^1 & \gamma^2 & \gamma^3 \\ h^1_* & 0 & 0 & 1 + l^1 h^1_* & 0 & 0 & 2l^1 + (l^1)^2 h^1_* & 0 & 0 \\ 0 & h^2_* & 0 & 0 & 1 + l^2 h^2_* & 0 & 0 & 2l^2 + (l^2)^2 h^2_* & 0 \\ 0 & 0 & h^3_* & 0 & 0 & 1 + l^3 h^3_* & 0 & 0 & 2l^3 + (l^3)^2 h^3_* \\ l^1 & -l^2 & 0 & \dfrac{(l^1)^2}{2} & -\dfrac{(l^2)^2}{2} & 0 & \dfrac{(l^1)^3}{3} & -\dfrac{(l^2)^3}{3} & 0 \\ l^1 & 0 & -l^3 & \dfrac{(l^1)^2}{2} & 0 & -\dfrac{(l^3)^2}{2} & \dfrac{(l^1)^3}{3} & 0 & -\dfrac{(l^3)^3}{3} \end{bmatrix}.$$

Setting $\Lambda(h^1_*, h^2_*, h^3_*) := \det[M(h^1_*, h^2_*, h^3_*)]$, we obtain the form (6.1). Also, by means of the precise computation by MAPLE, we can observe that $a_{123}, a_{12}, \cdots, a_0$ are positive. This completes the proof of (i).

Let us prove (ii). By using MAPLE, we can derive

$$\operatorname{rank}[M(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})] = 7.$$

This implies that the multiplicity of zero eigenvalues is equal to two, provided that $(h^1_*, h^2_*, h^3_*) = (h^1_{*,c}, h^2_{*,c}, h^3_{*,c}) \in \mathcal{S}$. Also, by using MAPLE again, we see that for $(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\}$

$$\operatorname{rank}[M(h^1_*, h^2_*, h^3_*)] = 8,$$

which means that the multiplicity of zero eigenvalues is equal to one, provided that $(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\}$. $\square$

Remark 6.2 *Set* $h^1_{*,a} := -a_{23}/a_{123}$. *Then, if* $h^1_* \neq h^1_{*,a}$, *the form of* $\Lambda(h^1_*, h^2_*, h^3_*) = 0$ *is represented as*

$$\left(h^2_* + \frac{a_{31}h^1_* + a_3}{a_{123}h^1_* + a_{23}}\right)\left(h^3_* + \frac{a_{12}h^1_* + a_2}{a_{123}h^1_* + a_{23}}\right) = \frac{(a_{31}a_{12} - a_{123}a_1)(h^1_* - h^1_{*,c})^2}{(a_{123}h^1_* + a_{23})^2} \tag{6.6}$$

with $a_{31}a_{12} - a_{123}a_1 \geq 0$. *Also, if* $h^1_* = h^1_{*,a}$, *the form of* $\Lambda(h^1_*, h^2_*, h^3_*) = 0$ *is represented as*

$$(a_{12}a_{23} - a_{123}a_2)h^2_* + (a_{23}a_{31} - a_{123}a_3)h^3_* + a_{23}a_1 - a_{123}a_0 = 0. \tag{6.7}$$

Note that $\Lambda(h^1_*, h^2_*, h^3_*) = 0$ *has the descriptions (6.6) and (6.7) if and only if the constants* $a_{123}, a_{12}, \cdots, a_0$ *satisfy*

$$4(a_2a_3 - a_{23}a_0)(a_{31}a_{12} - a_{123}a_1) - (a_{31}a_2 + a_{12}a_3 - a_{123}a_0 - a_{23}a_1)^2 = 0.$$

The other descriptions of $\Lambda(h^1_*, h^2_*, h^3_*) = 0$ *as (6.6) and (6.7) are derived by rotating the number* $\{1, 2, 3\}$ *in order.*

Let us analyze $\Lambda(h^1_*, h^2_*, h^3_*) = 0$ which implies that the operator $\mathcal{A}$ has zero eigenvalues. For simplicity, we only consider the case that

$$\gamma^1 = \gamma^2 = \gamma^3 = 1, \quad l^2 = l^3 = 1.$$

Note that $\gamma^1 = \gamma^2 = \gamma^3 = 1$ implies $\theta^1 = \theta^2 = \theta^3 = 2\pi/3$ by virtue of Young's law (1.8). Set $l^1 := d$. Then it follows that

$$\begin{aligned}
&a_{123} = 2\,d^4 + 16\,d^3 + 12\,d^2 + 4\,d + 2,\\
&a_{12} = a_{31} = 3\,d^4 + 32\,d^3 + 30\,d^2 + 12\,d + 7, \quad a_{23} = 8\,d^3 + 48\,d^2 + 24\,d + 4,\\
&a_1 = 48\,d^3 + 72\,d^2 + 36\,d + 24, \quad a_2 = a_3 = 12\,d^3 + 96\,d^2 + 60\,d + 12,\\
&a_0 = 36\,(1 + 2\,d)^2.
\end{aligned}$$

Also, $(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})$ is represented as

$$h^1_{*,c} = -\frac{d^2 + 12\,d\,(4\,d + 1)}{-1 + 6\,d^2 + 16\,d^3 + 3\,d^4}, \quad h^2_{*,c} = -3, \quad h^3_{*,c} = -3,$$

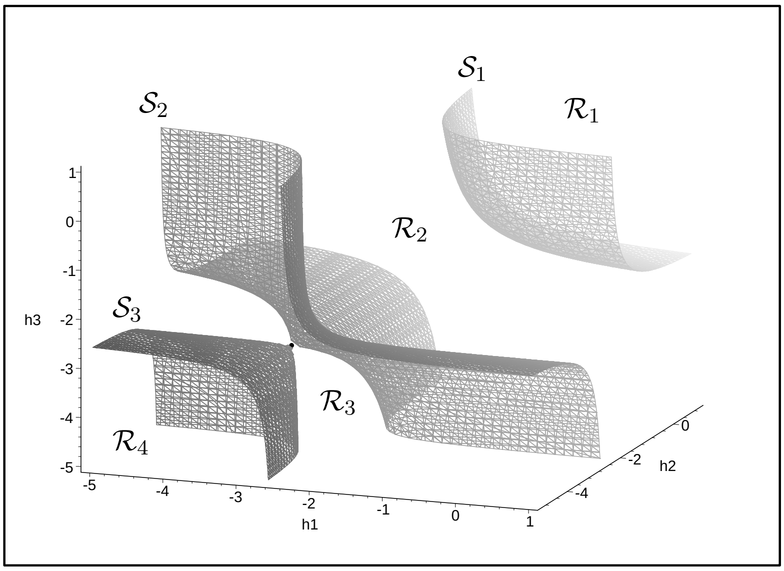

Figure 1: The configuration of $\mathcal{S}$; $d = 1\,(-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 > 0)$

which imply that the multiplicity of zero eigenvalues is equal to two. Further, we obtain

$$h^1_{*,a} = -\frac{a_{23}}{a_{123}} < 0, \quad h^2_{*,a} = -\frac{a_{31}}{a_{123}} < 0, \quad h^3_{*,a} = -\frac{a_{12}}{a_{123}} < 0 \tag{6.8}$$

and

$$h^1_{*,a} - h^1_{*,c} = \frac{2\,(1 + 6\,d + 3\,d^2 + 2\,d^3)^2}{(-1 + 6\,d^2 + 16\,d^3 + 3\,d^4)\,(d^4 + 8\,d^3 + 6\,d^2 + 2\,d + 1)},$$

$$h^2_{*,a} - h^2_{*,c} = h^3_{*,a} - h^3_{*,c} = \frac{-1 + 6\,d^2 + 16\,d^3 + 3\,d^4}{2\,(d^4 + 8\,d^3 + 6\,d^2 + 2\,d + 1)}.$$

It follows that $h^i_{*,a} > h^i_{*,c}\,(i = 1, 2, 3)$, provided that the parameter d is large enough such that $-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 > 0$. Then we obtain a situation as in Fig. 1. Also, it follows that $h^i_{*,a} < h^i_{*,c}\,(i = 1, 2, 3)$, provided that the parameter d is small enough such that $-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 < 0$. Then we are led to a situation as in Fig. 2. Now we prepare the following notations.

$$\begin{cases} \mathcal{S}_1 := \{(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \mid h^i_* > h^i_{*,a}\,(i = 1, 2, 3)\}, \\ \mathcal{S}_2 := \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_3), \\ \mathcal{S}_3 := \{(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \mid h^i_* < h^i_{*,a}\,(i = 1, 2, 3)\}, \end{cases} \tag{6.9}$$

$$\begin{cases} \mathcal{R}_1 := \{(h^1_*, h^2_*, h^3_*) \in \mathcal{R}_+ \mid h^i_* > h^i_{*,a}\,(i = 1, 2, 3)\}, \\ \mathcal{R}_2 := \mathcal{R}_- \setminus \mathcal{R}_4, \\ \mathcal{R}_3 := \mathcal{R}_+ \setminus \mathcal{R}_1, \\ \mathcal{R}_4 := \{(h^1_*, h^2_*, h^3_*) \in \mathcal{R}_- \mid h^i_* < h^i_{*,a}\,(i = 1, 2, 3)\}, \end{cases} \tag{6.10}$$

where $\mathcal{R}_+ := \{(h^1_*, h^2_*, h^3_*) \mid \Lambda(h^1_*, h^2_*, h^3_*) > 0\}$ and $\mathcal{R}_- := \{(h^1_*, h^2_*, h^3_*) \mid \Lambda(h^1_*, h^2_*, h^3_*) < 0\}$.

Theorem 6.3 *Let N_U be the number of positive eigenvalues and N_N be the number of zero eigenvalues. Set $\gamma^1 = \gamma^2 = \gamma^3 = 1$, $l^1 = d$, and $l^2 = l^3 = 1$.*

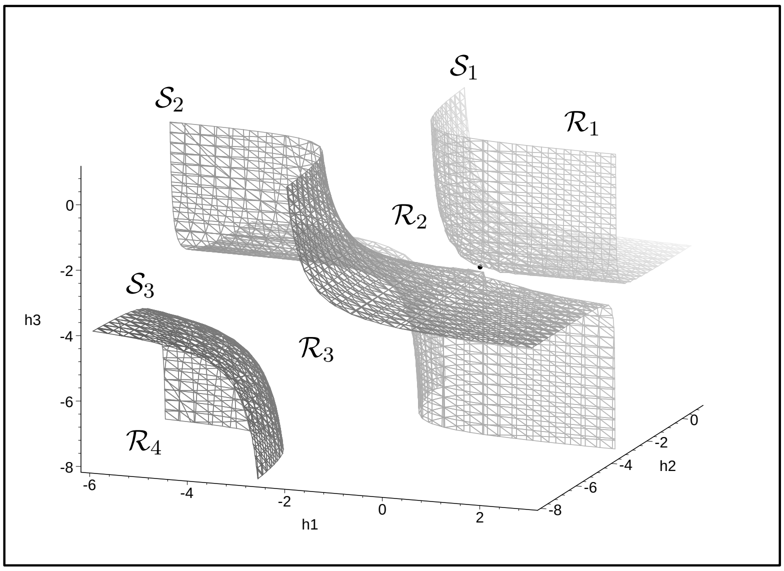

Figure 2: The configuration of $\mathcal{S}$; $d = 0.01\,(-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 < 0)$

(i) Assume that the parameter d satisfies $-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 > 0$. Then we obtain

$$
\begin{array}{lll}
N_U = 0,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_1 \quad (\textit{i.e. } \Gamma_* \textit{ is linearly stable}),\\
N_U = 0,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_1,\\
N_U = 1,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_2,\\
N_U = 1,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_2 \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\},\\
N_U = 1,\ N_N = 2 & \textit{if} & (h^1_*, h^2_*, h^3_*) = (h^1_{*,c}, h^2_{*,c}, h^3_{*,c}) \in \mathcal{S}_2 \cap \mathcal{S}_3,\\
N_U = 2,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_3,\\
N_U = 2,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_3 \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\},\\
N_U = 3,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_4,
\end{array}
$$

(ii) Assume that the parameter d satisfies $-1 + 6\,d^2 + 16\,d^3 + 3\,d^4 < 0$. Then we obtain

$$
\begin{array}{lll}
N_U = 0,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_1 \quad (\textit{i.e. } \Gamma_* \textit{ is linearly stable}),\\
N_U = 0,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_1 \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\},\\
N_U = 0,\ N_N = 2 & \textit{if} & (h^1_*, h^2_*, h^3_*) = (h^1_{*,c}, h^2_{*,c}, h^3_{*,c}) \in \mathcal{S}_1 \cap \mathcal{S}_2,\\
N_U = 1,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_2,\\
N_U = 1,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_2 \setminus \{(h^1_{*,c}, h^2_{*,c}, h^3_{*,c})\},\\
N_U = 2,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_3,\\
N_U = 2,\ N_N = 1 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_3,\\
N_U = 3,\ N_N = 0 & \textit{if} & (h^1_*, h^2_*, h^3_*) \in \mathcal{R}_4,
\end{array}
$$

Proof. We only prove (i). The proof of (ii) is given by a similar argument. Set $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$. Then this implies that

$$
I[\boldsymbol{v}, \boldsymbol{v}] = \int_0^d (v^1_\sigma)^2\, d\sigma + \int_0^1 (v^2_\sigma)^2\, d\sigma + \int_0^1 (v^3_\sigma)^2\, d\sigma \geq 0.
$$

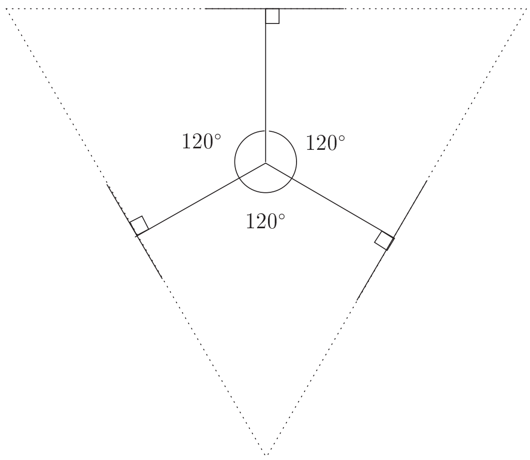

Figure 3: stable stationary solution ($d = 1$, $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$)

Since the maximal eigenvalue λ_1 allows the characterization

$$\lambda_1 = - \inf_{\boldsymbol{v} \in \mathcal{E} \setminus \{\boldsymbol{0}\}} \frac{I[\boldsymbol{v}, \boldsymbol{v}]}{(\boldsymbol{v}, \boldsymbol{v})_{-1}},$$

we have $\lambda_1 \le 0$. Also, it follows from Lemma 6.1(i) and $\Lambda(0, 0, 0) = 36\,(1 + 2\,d)^2 > 0$ that all eigenvalues are not zero for $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$. Thus, in this case, we see $\lambda_1 < 0$, so that all eigenvalues are negative. Further, by means of $(h^1_*, h^2_*, h^3_*) = (0, 0, 0) \in \mathcal{R}_1$, Lemma 5.7, and Lemma 6.1, we are led to $\lambda_1 < 0$ as long as $(h^1_*, h^2_*, h^3_*) \in \mathcal{R}_1$. Hence all eigenvalues are negative, provided that $(h^1_*, h^2_*, h^3_*) \in \mathcal{R}_1$. Using Lemma 5.7 and Lemma 6.1 again, we are led to the desired results. $\square$

Let us consider the stability of stationary solutions for situations as in Fig. 3 and Fig. 4. For a stationary solution as in Fig. 3, we observe (6.8) and

$$\Lambda(0, 0, 0) = a_0 = 36\,(1 + 2\,d)^2 > 0$$

for all $d > 0$. These imply that the point $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$ is included in $\mathcal{R}_1$ for all $d > 0$. Thus we obtain that all eigenvalues are negative, so that a stationary solution as in Fig. 3 is linearly stable. We remark that this agrees with the result which is shown in [11] in a sense of nonlinear. For a stationary solution as in Fig. 4, we derive

$$\Lambda(-1, -1, -1) = 0, \quad h^i_{*,a} = -\frac{7}{3} < -1\,(i = 1, 2, 3)$$

at $d = 1$. This gives that the point $(h^1_*, h^2_*, h^3_*) = (-1, -1, -1)$ exists on $\mathcal{S}_1$ in Fig. 1. Thus we obtain that one eigenvalue is zero and the others are negative, so that the stability of a stationary solution as in Fig. 4 is neutral in the linearized problem. We remark that the zero eigenvalue is a consequence of the fact that the total length is invariant under rotation in the geometry of Fig. 4.

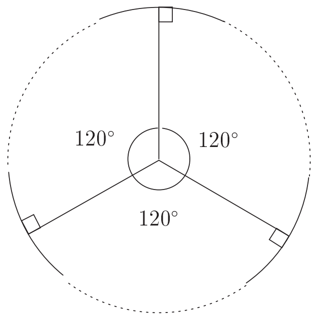

Figure 4: neutral stationary solution ($d = 1$, $(h^1_*, h^2_*, h^3_*) = (-1, -1, -1)$)

6.2 The case : $\kappa^1_* = 0$ and $\kappa^i \neq 0$ $(i = 2, 3)$

By means of $\kappa^1_* = 0$ and (2.1), we have $\gamma^2\kappa^2_* + \gamma^3\kappa_* = 0$. Set $\kappa^3_* := \kappa_*\,(\neq 0)$, which gives $\kappa^2_* = -\gamma^3\kappa_*/\gamma^2$. Now we only consider the case that

$$\gamma^1 = \gamma^2 = \gamma^3 = 1, \quad l^1 = \sqrt{3}\,, \quad l^2 = l^3\,(=: d).$$

Then we have $\kappa^2_* = -\kappa_*$. Under these assumptions, we obtain the following lemma.

Lemma 6.4 *(i) The operator $\mathcal{A}$ has zero eigenvalues if and only if (h^1_*, h^2_*, h^3_*) satisfies*

$$\widetilde{\Lambda}(h^1_*, h^2_*, h^3_*) = 0$$

with the form

$$\begin{aligned}\widetilde{\Lambda}(h^1_*, h^2_*, h^3_*) :=& \widetilde{a}_{123}h^1_*h^2_*h^3_* + \widetilde{a}_{12}h^1_*h^2_* + \widetilde{a}_{23}h^2_*h^3_* + \widetilde{a}_{31}h^3_*h^1_* \\ &+ \widetilde{a}_1h^1_* + \widetilde{a}_2h^2_* + \widetilde{a}_3h^3_* + \widetilde{a}_0,\end{aligned}$$

where $\widetilde{a}_{123}, \widetilde{a}_{12}, \cdots, \widetilde{a}_0$ are constants depending continuously on κ_* and d.

(ii) Set

$$\widetilde{\mathcal{S}} = \{(h^1_*, h^2_*, h^3_*)\,|\,\widetilde{\Lambda}(h^1_*, h^2_*, h^3_*) = 0\}.$$

The multiplicity of zero eigenvalues is equal to two if $(h^1_*, h^2_*, h^3_*) = (\widetilde{h}^1_{*,c}, \widetilde{h}^2_{*,c}, \widetilde{h}^3_{*,c}) \in \widetilde{\mathcal{S}}$, where $\widetilde{h}^i_{*,c}$ $(i = 1, 2, 3)$ are described by replacing $a_{123}, a_{12}, \cdots, a_0$ in (6.2) with $\widetilde{a}_{123}, \widetilde{a}_{12}, \cdots, \widetilde{a}_0$. Further, it is equal to one if $(h^1_*, h^2_*, h^3_*) \in \widetilde{\mathcal{S}} \setminus \{(\widetilde{h}^1_{*,c}, \widetilde{h}^2_{*,c}, \widetilde{h}^3_{*,c})\}$

Proof. In this case, eigenfunctions of zero eigenvalues have the form $v^1(\sigma) := \alpha^1_3\sigma^3 + \alpha^1_2\sigma^2 + \alpha^1_1\sigma + \alpha^1_0$ and $v^i(\sigma) := \alpha^i_1\sigma + \alpha^i_0 + \alpha^i_c\cos(\kappa^i_*\sigma) + \alpha^i_s\sin(\kappa^i_*\sigma)\,(i = 2, 3)$, where $\alpha^1_0, \alpha^1_1, \alpha^1_2, \alpha^1_3, \alpha^i_1, \alpha^i_0, \alpha^i_c, \alpha^i_s\,(i = 2, 3)$ are constants. A similar argument of the proof of Lemma 6.1 leads us to the desired results. Thus we omit it. $\square$

Let us analyze $\widetilde{\Lambda}(h^1_*, h^2_*, h^3_*) = 0$ which implies that the operator $\mathcal{A}$ has zero eigenvalues. For simplicity, we set

$$\kappa_* = \frac{1}{2}.$$

Then d can be chosen in

$$0 < d < \frac{8\pi}{3}.$$

Using MAPLE, it can be checked that $\widetilde{\Lambda}(h_*^1, h_*^2, h_*^3) = 0$ has the same descriptions as in Remark 6.2. Further, We can check by MAPLE that $\widetilde{h}_{*,c}^i < \widetilde{h}_{*,a}^i$ $(i = 1, 2, 3)$ are fulfilled and $\widetilde{h}_{*,a}^i$ $(i = 1, 2, 3)$ are monotone increasing with respect to d. Now we use the notations $\widetilde{\mathcal{S}}_j$ $(j = 1, 2, 3)$ and $\widetilde{\mathcal{R}}_k$ $(k = 1, 2, 3, 4)$, which are described by replacing $\Lambda(h_*^1, h_*^2, h_*^3)$ and $h_{*,a}^i$ in (6.9) and (6.10) with $\widetilde{\Lambda}(h_*^1, h_*^2, h_*^3)$ and $\widetilde{h}_{*,a}^i$, respectively.

Let us consider the stability of a stationary solution as in Fig. 5. According to [3], a stationary solution as in Fig. 5 with $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$ is stable. Also, MAPLE gives Fig. 8 as the configuration of $\widetilde{\Lambda}(h_*^1, h_*^2, h_*^3) = 0$ at $d = \pi/3$. Since

$$\begin{cases} \widetilde{\Lambda}(0, 0, 0) = \dfrac{3}{8} + \dfrac{\sqrt{3}\,\pi}{36} > 0, \\ \widetilde{h}_{*,a}^1 \approx -1.47 < 0, \quad \widetilde{h}_{*,a}^2 \approx -2.01 < 0, \quad \widetilde{h}_{*,a}^2 \approx -2.01 < 0 \end{cases}$$

at $d = \pi/3$, the point $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$ is included in $\widetilde{\mathcal{R}}_1$ in Fig. 8. Then it follows from Lemma 5.7 and Lemma 6.4 that a stationary solution as in Fig. 5 is stable, provided that $(h_*^1, h_*^2, h_*^3) \in \widetilde{\mathcal{R}}_1$. Further, we can obtain the same results as in Theorem 6.3 (i).

Let us consider the stability of stationary solutions as in Fig. 6 and Fig. 7. Note that $\widetilde{\Lambda}(h_*^1, h_*^2, h_*^3) = 0$ depends continuously on the parameter d with keeping the configuration as in Fig. 8 for $0 < d < 8\pi/3$. Then MAPLE gives Fig. 9 and Fig. 10 as the configurations of $\widetilde{\Lambda}(h_*^1, h_*^2, h_*^3) = 0$ at $d = 4\pi/3$ and $d = 2\pi$, respectively. We consider the stability of a stationary solution as in Fig. 6 with $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$. Since

$$\begin{cases} \widetilde{\Lambda}(0, 0, 0) = 0, \\ \widetilde{h}_{*,a}^1 \approx -0.67 < 0, \quad \widetilde{h}_{*,a}^2 \approx -0.44 < 0, \quad \widetilde{h}_{*,a}^2 \approx -0.44 < 0 \end{cases}$$

at $d = 4\pi/3$, the point $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$ exists on $\widetilde{\mathcal{S}}_1$ in Fig. 9. Thus we obtain that one eigenvalue is zero and the others are negative, so that the stability of a stationary solution as in Fig. 6 is neutral in the linearized problem. We remark that the zero eigenvalue is a consequence of the fact that the total length is invariant under translation in the geometry of Fig. 6. Finally, we consider the stability of a stationary solution as in Fig. 7 with $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$. Since

$$\begin{cases} \widetilde{\Lambda}(0, 0, 0) = -\dfrac{\pi^2}{2} - \dfrac{3\sqrt{3}\,\pi}{8} < 0, \\ \widetilde{h}_{*,a}^1 \approx -0.17 < 0, \quad \widetilde{h}_{*,a}^2 \approx -0.11 < 0, \quad \widetilde{h}_{*,a}^2 \approx -0.11 < 0 \end{cases}$$

at $d = 2\pi$, the point $(h_*^1, h_*^2, h_*^3) = (0, 0, 0)$ is included in $\widetilde{\mathcal{R}}_2$ in Fig. 10. Thus we obtain that one eigenvalue is positive and the others are negative, so that a stationary solution as in Fig. 7 is unstable.

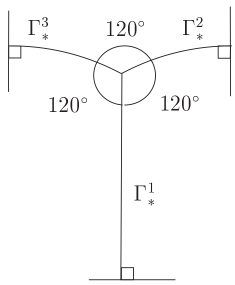

Figure 5: stable stationary solution ($d = \pi/3$, $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$)

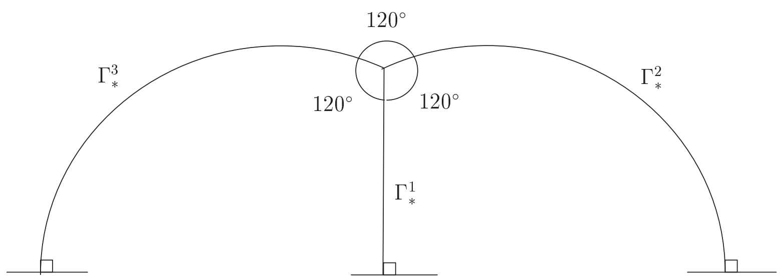

Figure 6: neutral stationary solution ($d = 4\pi/3$, $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$)

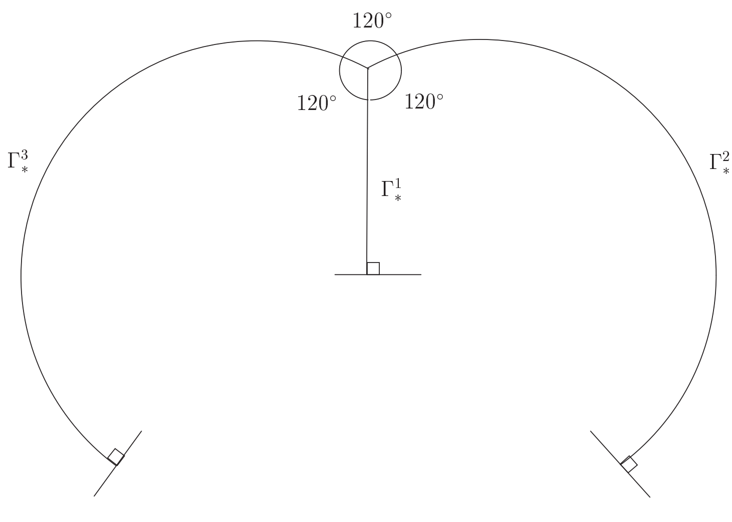

Figure 7: unstable stationary solution ($d = 2\pi$, $(h^1_*, h^2_*, h^3_*) = (0, 0, 0)$)

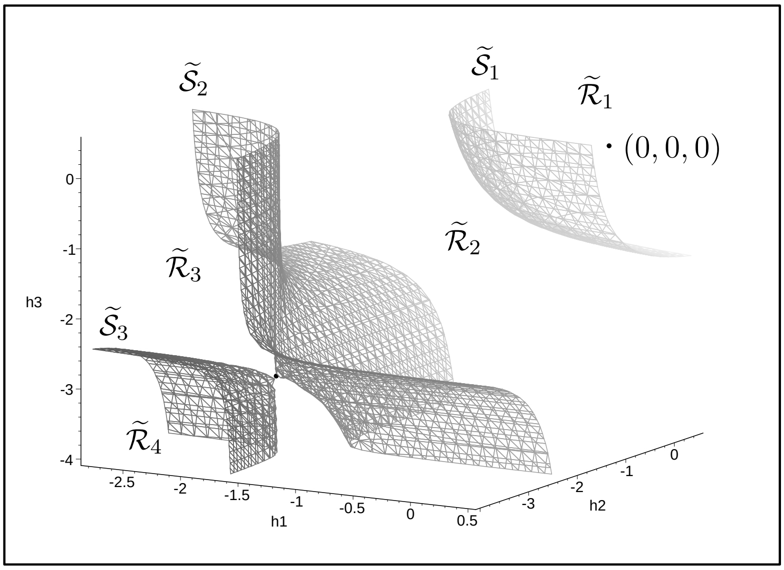

Figure 8: The configuration of $\widetilde{\mathcal{S}}$; $d = \pi/3$

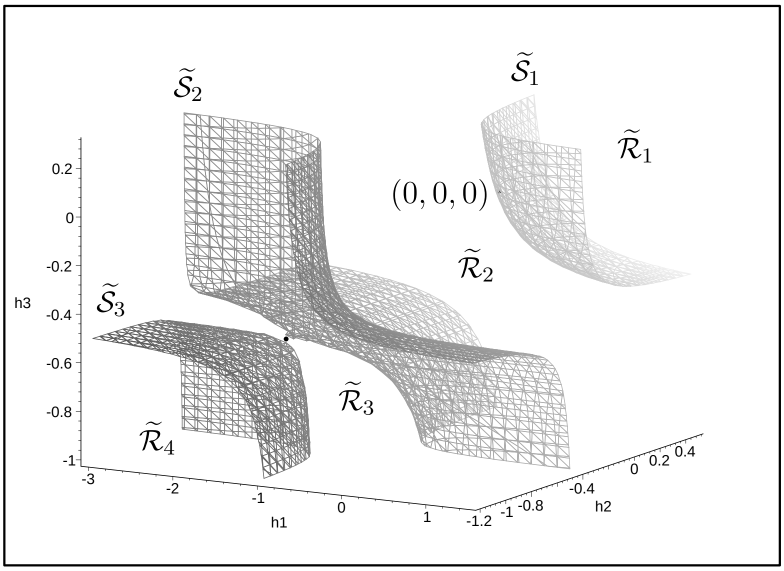

Figure 9: The configuration of $\widetilde{\mathcal{S}}$; $d = 4\pi/3$

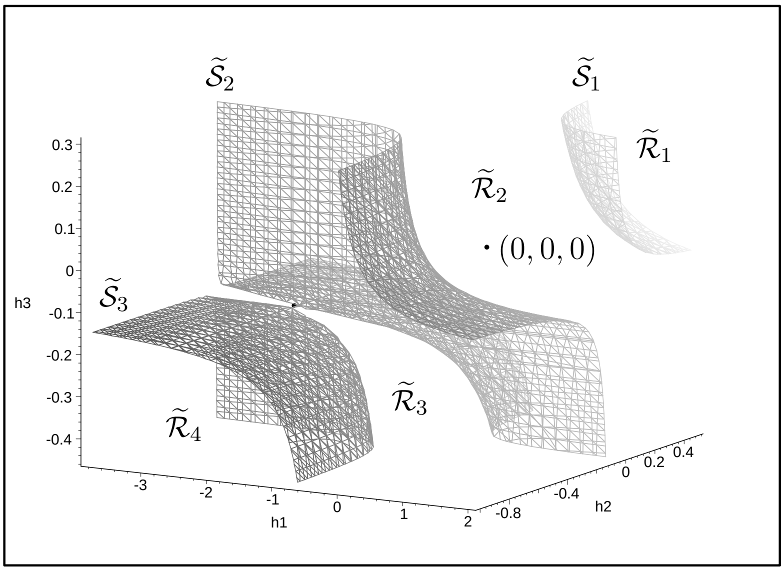

Figure 10: The configuration of $\widetilde{\mathcal{S}}$; $d = 2\pi$

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References

[1] Courant R. and Hilbert D., *Methods of mathematical physics*, vol.I, Interscience, New York, 1953.

[2] Elliott, C.M. and Garcke, H., *Existence results for diffusive surface motion laws*, Adv. Math. Sci. Appl. 7 (1997), No.1, 465–488.

[3] Escher, J., Garcke, H., and Ito, K., *Exponential stability for a mirror-symmetric three phase boundary motion by surface diffusion*, Math. Nachr., 257 (2003), 3–15.

[4] Escher, J., Mayer, U., and Simonett, G., *The surface diffusion flow for immersed hypersurfaces*, SIAM J. Math. Anal. 29 (1998), no. 6, 1419–1433.

[5] Ei, S.-I., Sato, M.-H., and Yanagida, E., *Stability of stationary interfaces with contact angle in a generalized mean curvature flow*, Amer. J. Math. 118 (1996), 653–687.

[6] Ei, S.-I. and Yanagida, E., *Stability of stationary interfaces in a generalized mean curvature flow*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40 (1993), 651–661.

[7] Garcke, H., Ito, K., and Kohsaka, Y., *Linearized stability analysis of stationary solutions for surface diffusion with with boundary conditions*, SIAM J. Math. Anal. 36 (2005), No. 4, 1031–1056.

[8] Garcke, H. and Novick-Cohen, A., *A singular limit for a system of degenerate Cahn-Hilliard equations*, Adv. in Diff. Equations 5 (2000), No. 4-6, 401–434.

[9] Hutchings, M., Morgan, F., Ritoré, M., and Ros A., *Proof of the double bubble conjecture*, Ann. of Math. 155 (2002), 459–489.

[10] Ikota, R. and Yanagida, E., *A stability criterion for stationary curves to the curvature-driven motion with a triple junction*, Differential Integral Equations, 16 (2003), No. (6), 707–726.

[11] Ito K. and Kohsaka Y., *Three phase boundary motion by surface diffusion in triangular domain*, Adv. Math. Sci. Appl. 11 (2001), 753–779.

[12] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[13] Zeidler, E., *Applied Functional Analysis: Applications to Mathematical Physics*, Springer-Verlag, New York, 1995.